# Week 6: Differential geometry and Killing vectors

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**Exercise 1**. *Embedded surfaces.* Consider a torus parametrised by the angles $(\theta, \phi)$ and embedded in flat three-dimensional Euclidean space. A possible parametrisation is

$$x = (R + r\sin\theta)\cos\phi\,, \qquad y = (R + r\sin\theta)\sin\phi\,, \qquad z = r\cos(\theta)\,. \tag{0.1}$$

1. Compute the metric on the torus, using $\theta$ and $\phi$ as coordinates on its surface.
2. Compute the Christoffel symbols of the Levi-Cività connection, the unique independent component of $R_{\mu\nu\sigma\rho}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar $R$.
3. Check that $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$.
4. Check that $\partial/\partial\phi$ is a Killing vector but that $\partial/\partial\theta$ is not. Interpret the result.

**Exercise 2**. *Killing vetors on the sphere.* Consider the metric of the two sphere with unit radius, $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$. Check that the vectors

$$\xi_{(1)} = \frac{\partial}{\partial\varphi}\,, \qquad \xi_{(2)} = \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\,, \qquad \xi_{(3)} = \sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}$$

are Killing vectors. Check that $\xi_{(k)} = \epsilon^{ijk}[\xi_{(i)}, \xi_{(j)}]$, with $\epsilon^{ijk}$ the *anti-symmetric symbol.* Argue that these vectors are related to rotations about the axes of $\mathbb{R}^3$.

**Exercise 3**. *Conical singularities.* Consider the surface of a cone whose angular aperture is $2\theta_0$ embedded in flat three-dimensional Euclidean space.

1. Write down the metric for this surface in terms of spherical coordinates.
   *Hint: Note that, on the cone, $\theta = \theta_0$.*
2. Consider the vector $U = \partial/\partial r$ at the point $(r, \phi) = (r_0, 0)$. What vector do we obtain if we parallel transport it from $\phi = 0$ to $\phi = 2\pi$ along the curve $\gamma_0$ defined by $r = r_0$?
3. Consider the integral curves of field $V = (\alpha, 1)$, where $\alpha > 0$ is a constant. In particular, consider the path $\gamma_1$ from $p_1 = (r_0, 0)$ to $p_2 = (r_0 + 2\pi\alpha, 2\pi)$ whose tangent vector is $V$. Compute the parallel transport of $U$ from $p_1$ to $p_2$ along $\gamma$.
4. (★★) Consider one more curve $\gamma_2$, whose tangent vector is proportional to $\partial/\partial\phi$ and goes from $p_2$ to $p_3 = (r_0 + 2\pi\alpha, 0)$. Finally, consider $\gamma_3$ going from $p_3$ to $p_0$ through the curve $\phi = 0$. Note that $\gamma_3 \circ \gamma_2 \circ \gamma_1$ is a closed path starting at $p_0$. Compute the parallel transport of $U$ along this path and compare it with what you obtained in the second item of this exercise. What does this result tell us about the curvature?

**Exercise 4.** (★★) *More on geodesics.* The length of a curve is given by

$$S = \int_{\tau_1}^{\tau_2} ||\gamma'(\tau)||\,\mathrm{d}\tau = \int_{\tau_1}^{\tau_2} \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,\mathrm{d}\tau\,,$$

where $\dot{f} = \mathrm{d}f/\mathrm{d}\tau$. Prove that geodesic curves extremise $S$. To do so, i) write down the Euler–Lagrange equations for the "Lagrangian" $L(x^\mu, \dot{x}^\mu) = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$, noting that $g_{\mu\nu}$ depends on $x^\alpha$ (but not on $\dot{x}^\alpha$), and ii) impose that $\tau$ is the affine parameter ($L(x^\mu, \dot{x}^\mu) = 1$).