Week 7: Trajectories in the Schwarzschild geometry

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Exercise 1. *Pound–Rebka experiment.* From the roof of a building, we send a laser beam towards its bottom. Compute the blue-shift experienced by light due to the effect of gravity in terms of the height h of the building, the gravitational acceleration on the surface of the Earth g, and the speed of light c. Recall that the frequency measured by and observer U^μ is $\hbar\omega = -K_\mu U^\mu$, with K^μ the quadri-momentum of the photon.
Hint: It suffices to use the expansion of the Schwarzschild metric around flat space.

Exercise 2. *Period of a circular orbit.* Consider a particle orbiting an object of mass M. If it is performing stable orbits around the object, what is its period measured in "coordinate time" t, in terms of which the Schwarzschild metric is written? Compare it with the Newtonian result.
Note: coordinate time t coincides with time measured by an observer standing still at infinity, where the metric flattens.

Exercise 3. *Relativistic effects in GPS systems.* GPS satellites fly in circular orbits at an altitude of 20200 km. The most modern GPS systems goal is to be able to deliver centimeter-level positions, so that they can be used for autonomous driving, for example. For that, they need to be precise to the level of ~ 20 nanoseconds. You may want to know more by visiting this webpage.

- Using Newtonian gravity, compute the period T of the orbit of a GPS satellite, its orbital velocity v and the ratio $\hat{L}$ between its angular momentum and its mass.
- Estimate the order of magnitude of the maximum velocity of particles moving on the surface of the Earth. Argue that it is a sensible assumption to consider these particles to be approximately at rest. Do not forget that the Earth is rotating.
- Compare the ratio between the frequencies of signals emitted from the satellite and received on Earth. Conclude that, after one day, the clocks in the satellites will be off by ~ 38 microseconds. Will they be advanced or retarded?

Here is some data that you will need:

$$R_\oplus = 6.37 \times 10^3\,\mathrm{km}\,(\textit{Earth radius}),\;\; M_\oplus = 5.97 \times 10^{24}\,\mathrm{kg}\,(\textit{Earth mass}),$$
$$c = 3.00 \times 10^8\,\mathrm{m\,s^{-1}}\,(\textit{speed of light}),\;\; \mathrm{G} = 6.67 \times 10^{-20}\,\mathrm{km^3\,kg^{-1}\,s^{-2}}\,(\textit{Newton's constant}).$$

Exercise 4. (★★) *Measuring a mass by going around it.* A spacecraft follows a circular geodesic around a spherical object of mass M. Let S^μ be a quadri-vector parallel transported along the geodesic and such that $S_\mu U^\mu = 0$, where U is the quadri-velocity of the spacecraft. In this situation, the spatial components of S^μ describe the movement of the axes of a gyroscope inside the spacecraft. Assume that at the beginning of a given orbit, this object points outward and in the radial direction (toward a distant star, for example). Find M in terms of the angle between the radial direction and the gyroscope axes after one orbit.