Week 8: Black holes and how to approach them

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Exercise 1. *Curvature singularities and geodesic deviation.*

1. Write down the Christoffel symbols of the Schwarzschild metric in terms of the coordinates (t, r, θ, φ) as defined in the notes.
   Hint: You do not have to compute them again, just substitute the solutions for $\phi(r, t)$ and $\lambda(r, t)$ in the expressions we computed in class (and are written in the notes).

2. From the previous, check that for the Schwarzschild metric,

$$R^t{}_{rtr} = 2\frac{M}{r^3}\left(1 - \frac{2M}{r}\right)^{-1}, \qquad R^t{}_{\theta t\theta} = (\sin\theta)^{-2} R^t{}_{\varphi t\varphi} = -\frac{M}{r},$$
$$R^\theta{}_{\varphi\theta\varphi} = \frac{2M}{r}\sin^2\theta, \qquad R^r{}_{\theta r\theta} = (\sin\theta)^{-2} R^r{}_{\varphi r\varphi} = -\frac{M}{r}.$$

   Even though you are not required to prove it, note that the rest of components of the Riemann tensor are either determined in terms of them or zero.

3. Find the forms $\{\omega^{\hat{t}}, \omega^{\hat{r}}, \omega^{\hat{\theta}}, \omega^{\hat{\varphi}}\}$ proportional to $\{\mathrm{d}t, \mathrm{d}r, \mathrm{d}\theta, \mathrm{d}\varphi\}$ respectively, in terms of which the metric reads

$$\mathrm{d}s^2 = -\omega^{\hat{t}} \otimes \omega^{\hat{t}} + \omega^{\hat{r}} \otimes \omega^{\hat{r}} + \omega^{\hat{\theta}} \otimes \omega^{\hat{\theta}} + \omega^{\hat{\varphi}} \otimes \omega^{\hat{\varphi}}.$$

   Interpret them as the *static local frame* in the Schwarzschild metric. The corresponding basis vectors fulfil $\omega^j(\mathbf{e}_i) = \delta^j_i$. Check that, in this frame,

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -\frac{2M}{r^3}, \quad R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\varphi}\hat{t}\hat{\varphi}} = \frac{M}{r^3}, \quad R_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}} = \frac{2M}{r^3}, \quad R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\varphi}\hat{r}\hat{\varphi}} = -\frac{M}{r^3}.$$

4. (★★) If $\mathbf{e}_{\hat{\rho}}$ is the infalling observer's radial basis and $\mathbf{e}_{\hat{\tau}} = \mathrm{d}/\mathrm{d}\tau$ is their time basis vector, show that in the infalling observer's local inercial frame the components of the Riemann tensor remain unchanged;

$$R_{\hat{\tau}\hat{\rho}\hat{\tau}\hat{\rho}} = -\frac{2M}{r^3}, \quad R_{\hat{\tau}\hat{\theta}\hat{\tau}\hat{\theta}} = R_{\hat{\tau}\hat{\varphi}\hat{\tau}\hat{\varphi}} = \frac{M}{r^3}, \quad R_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}} = \frac{2M}{r^3}, \quad R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\varphi}\hat{\rho}\hat{\varphi}} = -\frac{M}{r^3}.$$

   Note that in this frame, freely moving particles momentarily at rest are separated by the 3-vector $\xi = \xi^{\hat{j}}\mathbf{e}_{\hat{j}}$. Compute $\frac{\mathrm{D}^2\xi^{\hat{j}}}{\mathrm{d}\tau^2}$ in this basis and interpret the result in terms of *spaghettification*.

5. Compute the Kretschmann scalar, $K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. What happens to curvature invariants when $r = 2M$? And at $r = 0$? Comment also on tidal forces experienced across the horizon.

Exercise 2. *The Interstellar black hole [Spoiler alert].* The black hole Gargantua, from the movie *Interstellar*, has a mass of $M = 10^8 M_\odot$ (with $M_\odot$ is the mass of the Sun). Miller's planet orbits circularly around it, and its period is 1.7 hours for an observer at infinity. Conclude that the spacetime in the movie is not spherically symmetric.