Week 10: Cosmology

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Exercise 1. In a FLRW universe, show that $U = \partial/\partial t$ is not a Killing vector, and thus it make sense to say that the energy of individual particles is not conserved. After that,

- prove that the tensor $K_{\mu\nu} = a^2 (g_{\mu\nu} + U_\mu U_\nu)$ is a Killing tensor (i.e. $\nabla_{(\rho} K_{\mu\nu)} = 0$);
- check that, if V is a geodesic vector, $K(V, V) = K_{\mu\nu} V^\mu V^\nu$ is a conserved quantity along the integral curve of V and
- show that, if V is the quadri-velocity of a galaxy following a geodesic, its spacial velocity as measured by observers who see the Universe isotropic and homogeneous decays as $V_i V^i \propto a^{-2}$.

Exercise 2. Show that in a FLRW universe where there is only vacuum energy the Ricci scalar is constant and that the Ricci tensor is proportional to the metric.

Exrcise 3. *(Einstein's major mistake)*. We have seen that the Universe expands and, quite generically, sensible equations of state for its content imply the existence of a initial singularity, that we call the Big Bang. In 1917, Einstein introduced the *cosmological constant* Λ for the first time, driven by the prejudice that the Universe was static.

- Consider a closed FLRW universe filled with (non-relativistic) matter and with a cosmological constant Λ. From the acceleration equation, find the value of the matter density $\rho_m = \rho_{m,0}$ in terms of the cosmological constant (or, equivalently, vacuum density ρ_Λ) that makes $\ddot{a} = 0$.
- Using the Friedmann equation, find the value of $a = a_0$ for which the universe is static.

(★★) However, this universe is actually unstable. Consider a homogeneous small perturbation of the matter density $\rho_m(m) = \rho_{m,0} + \epsilon\delta\rho(t) + \mathcal{O}(\epsilon^2)$, which may induce a perturbation of the scale factor $a(t) = a_0 + \epsilon\delta a(t) + \mathcal{O}(\epsilon^2)$.

- From the acceleration equation, find an equation for $\delta a''(t)$ as a function of $\delta\rho(t)$. From the Friedmann equation, find an equation for $\delta\rho'(t)$ as a function of $\delta a'(t)$.
- Perform a change of variables $\delta a'(t) = a_0 f_1(t)/3$ and $\delta\rho(t) = f_2(t)/(4\pi)$ (note we are setting $G = c = 1$). Show that the two independent solutions of the system are $f_i^\pm(t) = A_i^\pm \exp(\pm 2t\sqrt{2\pi\rho_\Lambda})$. Find the relation between the integration constants $A_1^\pm$ and $A_2^\pm$ for each choice of the sign.
- Conclude that Eisntein's static universe is unstable, in the sense that any small perturbation will grow exponentially either to the future or to the past.

Hubble's observations of the expansion of the universe in 1929 proved Einstein wrong. After his death, the physicist Gueorgui Gámov declared that Einstein admitted "that the cosmic repulsion idea was the biggest blunder he had made in his entire life." Ironically, since 1998 we know the expansion is actually accelerating, so Λ is included again in our cosmological model.