Week 11: Black hole thermodynamics and finite temperature AdS/CFT[1]

Javier Subils, javier.subils@su.se. *February 3, 2026*

In the semicalssical approximation to Euclidean quantum gravity path integral, the thermodynamical partition function and the free energy are given by

$$\mathcal{Z}[\beta] = e^{\beta F} \simeq e^{I_E[g^{(\text{cl})}]} , \tag{0.1}$$

where $I_E[g^{(\text{cl})}]$ is the Euclidean action of a classical solution $g^{(\text{cl})}$ with periodic boundary conditions in imaginary time. Often, this action is divergent and needs renormalization. A way to do it is by subtracting the action $I_E[g^{(0)}]$ of a reference spacetime $g^{(0)}$ with the same asymptotics. With this approach, we will compute the study the thermodynamics of the Schwarschild black hole and of a black brane in AdS.

Exercise 1. (*Schwarschild thermodynamics*) For the Euclidean Schwarschild solution

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right) \mathrm{d}\tau^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2 , \tag{0.2}$$

the natural background metric $g^{(0)}$ to consider is the Euclidean four-dimensional flat space (i.e., Wick-rotated Minkowski solution). This can be regarded as the ground state of asymtotically flat spacetimes. On the other hand, the Euclidean action is

$$I_E = -\frac{1}{16\pi G} \int_M \mathrm{d}^4x \sqrt{g} R - \frac{1}{8\pi G} \int_{\partial M} \mathrm{d}^3x \sqrt{h} K . \tag{0.3}$$

For vacuum solutions, $R = 0$ and then the Einstein-Hilbert action term in the action vanishes. It is thus the Gibbons–Hawking term that contributes, so

$$I_E[g^{(\text{cl})}] - I_E[g^{(0)}] = -\frac{1}{8\pi G} \int_{\partial M} \mathrm{d}^3x \sqrt{h} (K - K^{(0)}) . \tag{0.4}$$

For this expression to be consistent, it is necessary that the induced geometries on the boundaries ∂M are the same, $h = h^{(0)}$. However, the extrinsic curvatures will still be different, and this is what gives rise to a non-zero value of the renormalised action.

- Take the boundary hypersurface ∂M to be the sphere with large but finite radius $r = R_b$, which we will eventually send to infinity. Write down the boundary metrics of Euclidean Schwarschild and flat Euclidean spacetimes.

- Note that the angular parts are already equal. Fix the period of the background temporal direction β_0 in terms of the period of the black hole solution β, so that the circles of Euclidean time are also the same in both geometries,

$$\int_0^{\beta} \mathrm{d}\tau \sqrt{h_{\tau\tau}} = \int_0^{\beta_0} \mathrm{d}\tau \sqrt{h^{(0)}_{\tau\tau}} . \tag{0.5}$$

- Compute the integral (0.4). Use $\sqrt{h}K = n^\mu \partial_\mu \sqrt{h}$, with n the radial unit normal (*Hint: $n \propto \mathrm{d}r$ in both geometries*). After performing the integrals, take the limit $R_b \to \infty$ and identify the free energy relative to Minkowski ground state.

[1]Based on an exercise sheet from Prof. Roberto Emparan, Universitat de Barcelona.

- Expressing F as a function of the temperature, use conventional thermodynamics

$$E = \frac{\partial(\beta F)}{\partial \beta}\,, \qquad S = \beta \frac{\partial(\beta F)}{\partial \beta} - \beta F\,, \tag{0.6}$$

  to obtain the energy E and entropy S of a Schwarschild black hole. Check that you get the expected result.

Exercise 2. (*AdS black brane thermodynamics.*) Now consider the five-dimensional Euclidean geometry

$$\mathrm{d}s^2 = \frac{r^2}{L^2}\left(f(r)\mathrm{d}\tau^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right) + \frac{L^2}{r^2}\frac{\mathrm{d}r^2}{f(r)}\,, \qquad f(r) = 1 - \frac{r_\mathrm{H}^4}{r^4}\,, \tag{0.7}$$

which is a solution to Einstein's equations with a negative cosmological constant $\Lambda = -6/L^2$. In particular, $R_{\mu\nu} = -4g_{\mu\nu}/L^2$. This solution is derived from the action

$$I_E = -\frac{1}{16\pi G}\int_M \mathrm{d}^4x\sqrt{g}\left(R + \frac{12}{L^2}\right) - \frac{1}{8\pi G}\int_{\partial M}\mathrm{d}^3x\sqrt{h}K\,. \tag{0.8}$$

In (0.7), the radial coordinate extends from $r = r_\mathrm{H}$ to $r \to \infty$, with r_H a constant where the *blackening factor* $f(r)$ vanishes. The *AdS radius* L is also a constant with dimensions of length. The Lorentzian version of (0.7) is a black hole geometry with an event horizon at $r = r_\mathrm{H}$. This black hole extends in the three x^i directions; thus, it is often referred to as a *black brane.* Let the (infinite) volume of these three dimensions be $V := \int \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$. Due to translational invariance along them, quantities like the total mass (or energy) $M = E$, entropy S, free energy F..., are extensive (i.e. proportional to V). For this reason, we will consider the corresponding densities, such as the energy density $\rho = E/V$, the entropy density $s = S/V$, and the free energy density $f = F/V$.

- Expanding the metric about $r = r_\mathrm{H}$, where the metric approaches that of Euclidean Rindler space, obtain the temperature T of the black brane.

- From the Euclidean action in (0.8), show that the free energy density is

$$f = -\frac{\pi^3 L^3}{16G}T^4\,. \tag{0.9}$$

- Now, compute the energy and the entropy densities. Check that the latter coincides with the 'area density' of the horizon over $4G$. If the pressure is given by $P = -f$, check the conventional thermodynamic relation

$$E + PV = TS\,. \tag{0.10}$$

- Compute the specific heat $c = \mathrm{d}E/\mathrm{d}T$. Is it positive or negative?

Observe that $\rho \propto T^4$, $s \propto T^3$. A gas of photons satisfies the same relations. Actually, any scale-invariant thermal system in 3+1 dimensions would. In particular, note that $P = \rho/3$, which is the equation of state that we used for a radiation dominated universe.

A five-dimensional black brane in AdS fulfilling the same thermodynamic relations as a conformally invariant four-dimensional fluid is one of the most basic features of the AdS/CFT correspondence.