General Relativity

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These set of lectures notes have been prepared for the EÐL610M General Relativity course (Spring 2023), delivered mostly online at the University of Iceland; within the *Theoretical Physics Specialisation Program*, in partnership with Nordita. They focus on the second part of the course. For this reason, all the relevant concepts of differential geometry are assumed and focus is placed on the physics.

Updates of these notes may be found in this webpage.

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Contents

1 Introduction

In this part of the course, we will focus on finding solutions to Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor of the spacetime metric $g_{\mu\nu}$, R stands for its trace, and $T_{\mu\nu}$ corresponds to the energy-momentum tensor. We will be working in geometrised units, $G = c = 1$.

It is understandable to feel somehow uneasy the first time one encounters Eqs. (1.1) and wonders about their meaning. Indeed, if read literally, this innocent expression is telling nothing less than

$$\text{space} \;=\; \text{matter}. \tag{1.2}$$

Anyway, we do not have time to further contemplate the depth of what these equations are saying. Rather, as physicists, we will attempt to find solutions to them and model systems and phenomena that we encounter in our Universe. But let us not forget that, behind the formulae we will be dealing with, we will be trying to describe not only the properties of those little white dots that bright upon our heads in clear and freezing winter Nordic nights, but also many other that we do not see with the naked eye and yet they are there. After all, we attempt to do something that comes with human nature, which is to dare to ask: "What is there around us?" And even, "Where do we come from?"

I hope these lectures give you the tools to approach these deep questions from the physicist point of view. Whether they foster deeper exploration, that is up to you. But,

> *Do not go gentle into that good night,*
> *Old age should burn and rave at close of day;*
> *Rage, rage against the dying of the light.*
>
> *Though wise men at their end know dark is right,*
> *Because their words had forked no lightning they*
> *Do not go gentle into that good night.*
>
> *Rage, rage against the dying of the light.* [1]

[1] From The Poems of Dylan Thomas, published by New Directions. This is the poem that is recited in the movie *Interstellar* while they start heading towards Saturn.

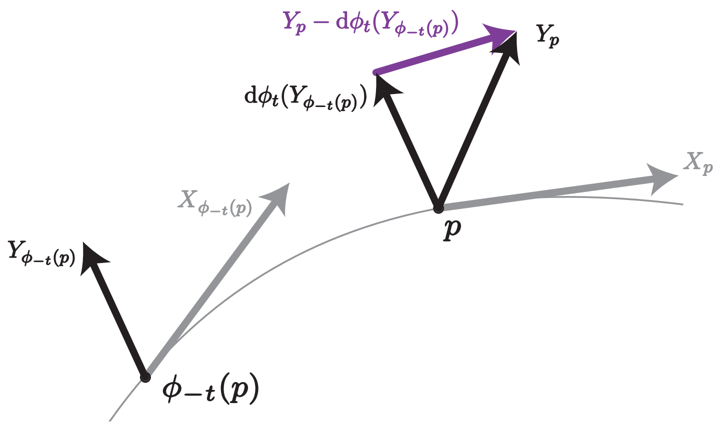

Figure 1: Pictorial representation of the Lie derivative of a vector Y with respect to a vector X, namely $\pounds_X Y$, before taking the limit $t \to 0$.

2 Differential geometry and conventions

Before starting these set of lectures, it is convenient to fix the notation we will be using. We are of course following [1], as in the first part of the course. This will also serve as a summary of what you should have learned so far.

2.1 Lie derivative and Lie bracket

Let M be a differential manifold and $\mathscr{X}(M)$ the set of its vector fields. When we are interested in defining "derivatives" of vector fields, we encounter the issue that we have to compare vector fields evaluated in two different points of the manifold, though eventually we want to take the limit in which both points approach each other. The problem is that when evaluated at different points, we obtain objects that belong to two different tangent spaces, and there is not a unique way to relate them.

For example, one option is to use the **integral curves** of vector fields themselves. For every vector field $X \in \mathscr{X}(M)$, we can define a function

$$\begin{aligned} \Phi :\ \mathbb{R} \times M &\longrightarrow M \\ (t,p) &\mapsto \Phi(t,p) \end{aligned} \tag{2.1}$$

such that, for each p, $\gamma(t) := \Phi(t,p)$ is an integral curve of X. This is to say that the derivative of the curve coincides with the vector field at every point on the curve, $\gamma'(t) = X_{\gamma(t)}$; and on $\gamma(0) = p$. In other words, for every point in M, the function Φ gives a path starting at p and whose tangent vector is X.

Moreover, for each value of t Eq. (2.2) defines a function from (an open subset U of) M to M,

$$\begin{aligned} \phi_t :\ U &\longrightarrow M \\ p &\mapsto \phi_t(p) \end{aligned} . \tag{2.2}$$

This function ϕ_t is differentiable by construction, and consequently defines a map between tangent spaces at two points on M

$$\begin{aligned} \mathrm{d}\phi_t : T_{\phi_{-t}(p)}M &\longrightarrow T_pM \\ Y_{\phi_{-t}(p)} &\mapsto \mathrm{d}\phi_t(Y_{\phi_{-t}(p)}) \end{aligned} \tag{2.3}$$

such that $\mathrm{d}\phi_t(Y_{\phi_{-t}(p)})(f) := Y_{\phi_{-t}(p)}(f \circ \phi_t)$. Then, the Lie derivative at every point p is defined as (see Fig. 1)

$$(\pounds_X Y)_p = \lim_{t\to 0} \frac{1}{t}\Big(Y_p - \mathrm{d}\phi_t(Y_{\phi_{-t}(p)})\Big) \in T_pM\,, \tag{2.4}$$

and this defines a vector field $\pounds_X Y \in \mathscr{X}(M)$. It can be proven that, in coordinates,

$$(\pounds_X Y)^i = \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right) = -(\pounds_Y X)^i\,. \tag{2.5}$$

Quite often, it is also called as **Lie bracket**, and denoted by $\pounds_X Y = [X, Y]$. Finally, let us mention that the definition of the Lie derivative of a vector in Eq. (2.4) can be generalised to any tensor. The coordinates of the **Lie derivative of a tensor** T with respect to a vector field X are given by

$$\begin{aligned} (\pounds_X T)^{i_1\cdots i_p}_{j_1\cdots j_q} = X^s \frac{\partial T^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial x^s} \; + \; T^{i_1\cdots i_p}_{k\, j_2\cdots j_q}\frac{\partial V^k}{\partial x^{j_1}} \; + \; \cdots \; + \; T^{i_1\cdots i_p}_{j_1\cdots j_{q-1} k}\frac{\partial V^k}{\partial x^{j_q}} \\ - \; T^{k\, i_2\cdots i_p}_{j_1\cdots j_q}\frac{\partial V^{i_1}}{\partial x^k} \; - \; \cdots \; - \; T^{i_1\cdots i_{p-1}\, k}_{j_1\cdots j_q}\frac{\partial V^{i_p}}{\partial x^k}\,. \end{aligned} \tag{2.6}$$

The Lie derivative is a quite interesting tool, as it measures how vectors change along the integral curves of other vectors. Yet, we shall define another type of derivative, the connection, which we examine in the next section.

2.2 Connections and covariant derivatives

In a curved manifold M, we would like to mimic the notion of "derivative with respect to a direction" or directional derivative, found in flat space. In a curved space this will become the *covariant derivative*. If $\mathcal{D}$ was a such thing, for two vector fields $X, Y \in \mathscr{X}(M)$ and any differential map $f \in \mathcal{F}(M)$ we would like it to satisfy

$$\mathcal{D}_{fX} Y = f \mathcal{D}_X Y \tag{2.7}$$

The intuition behind this requirement is that we are promoting the $\mathbb{R}$-linearity with respect to the first coordinate encountered in directional derivatives to $\mathcal{F}(M)$-linearity in all the manifold for covariant derivatives. Put differently, the adjective *covariant* signifies that local rescalings of a vector field X just rescale locally by the same amount the derivative with respect to X. Note that the Lie derivative fails to accomplish this property generically, as

$$\pounds_{fX} Y = f \pounds_X Y - Y(f) X \neq f \pounds_X Y\,, \tag{2.8}$$

which is only true when f is a constant function on M. A **connection** is precisely a map

$$\begin{aligned} \nabla : \mathscr{X}(M) \times \mathscr{X}(M) &\longrightarrow \mathscr{X}(M) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}\,. \tag{2.9}$$

that satisfies the property in Eq. (2.7),

i) $\nabla_{fX} Y = f \nabla_X Y$.

together with the expected properties of any derivative,

ii) $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_1} Y$,

iii) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$ and

iv) $\nabla_X (fY) = X(f) Y + f \nabla_X Y$.

This does not uniquely define the connection. However, it implies that a connection will be characterised by how it acts on the coordinate vectors,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} \,. \tag{2.10}$$

The coefficients Γ^k_{ij} are known as the Christoffel symbols. Importantly, they do not generically transform as a tensor. The **covariant derivative** of a vector field U with respect to the vector field V then becomes

$$\nabla_U V = \left[U^i \partial_i V^k + \Gamma^k_{ij} U^i U^j \right] \frac{\partial}{\partial x^k} \,. \tag{2.11}$$

Given a Lorentzian differential manifold (M, g) (where M is a differential manifold and $g = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$ is a Lorentzian metric defined on it), there is a unique connection ∇ that i) is compatible with the metric, $\nabla_X g = 0$ for any vector X and ii) is symmetric (*i.e.* torsionless), $\Gamma^k_{ij} = \Gamma^k_{ji}$. This connection is known as the **Levi-Cività connection** and its Christoffel symbols are given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu} \right) \,. \tag{2.12}$$

What is the difference between the Lie derivative $\pounds_X Y$ and the covariant derivative $\nabla_X Y$? The most evident one, is that the Lie derivative of the coordinate vectors is identically zero, $\pounds_{\partial_i} \partial_j = 0$; whereas $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$. In particular, when ∇ is the Levi-Cività connection, it cares about the metric whereas $\pounds$ does not (actually, the Lie derivative can be defined in manifold without a metric). Note, however, that

$$\nabla_X Y - \nabla_Y X = \left[X^i \partial_i Y^k + \Gamma^k_{ij} X^i Y^j \right] \frac{\partial}{\partial x^k} - \left[Y^i \partial_i X^k + \Gamma^k_{ij} Y^i X^j \right] \frac{\partial}{\partial x^k} = [X, Y] \,, \tag{2.13}$$

which turns out to be true for any symmetric connection.

2.3 Killing vectors and geodesics

We say that a vector field $\xi \in \mathscr{X}(M)$ is a **Killing vector field** if the Lie derivative of the metric with respect to it is zero, $\pounds_\xi g = 0$. Since the metric is a tensor, from Eq. (2.6) we can get the equation that any Killing vectors must obey,

$$\xi^k \partial_k g_{ij} + g_{kj} \partial_i \xi^k + g_{ik} \partial_j \xi^k = 0 \,. \tag{2.14}$$

Additionally, the fact that the connection is compatible with the metric implies (for example, from Eq. (2.12)) the relation $\partial_k g_{ij} = \Gamma^l_{ik} g_{lj} + \Gamma^l_{lj} g_{il}$; which substituted into the Killing Eq. (2.14) permits to rewrite it as

$$\nabla_{(i}\xi_{j)} = 0\,. \tag{2.15}$$

Finally, recall that we say that a curve $\gamma : I \to M$ (with $I \subset \mathbb{R}$) is **geodesic** if its tangent vector $\gamma' = \mathrm{d}/\mathrm{d}\tau$ is transported parallelly along the curve, $\nabla_{\gamma'}\gamma' = 0$; in coordinates

$$V^i \nabla_i V^j = 0\,. \tag{2.16}$$

In particular, $||\gamma'||^2 := g(\gamma', \gamma')$ is constant along the curve and $\gamma(t) = (x^1(t), \ldots, x^n(t))$ fulfills the geodesic equations,

$$\frac{\mathrm{d}^2 x^k}{\mathrm{d}\tau^2} + \Gamma^k_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}\tau} \frac{\mathrm{d}x^j}{\mathrm{d}\tau} = 0\,. \tag{2.17}$$

Note that, if $\Gamma^k_{ij} = 0$, the metric is flat and the geodesic equations lead to uniform linear motion. Additionally, for every Killing vector ξ the quantity $\xi_i V^i = g_{ij} V^i \xi^j$ will be conserved along the curve, since

$$V^i \nabla_i (V^k \xi_k) = V^i V^k \nabla_{(i}\xi_{k)} + \xi_j V^i \nabla_i V^j = 0\,, \tag{2.18}$$

the two terms being zero due to the Killing equation Eq. (2.15) and geodesic motion Eq. (2.16), respectively.

2.4 Curvature

Given a connection ∇, the **Riemann tensor** is defined as

$$\begin{aligned} R : \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) &\longrightarrow \mathscr{X}(M) \\ (X, Y, Z) &\mapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,. \end{aligned} \tag{2.19}$$

If ∇ is the Levi-Cività connection, the components of the Riemann tensor read[2]

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}\,. \tag{2.20}$$

It is also useful to consider the tensor obtained when lowering the first index, $R_{\mu\nu\rho\sigma} = g_{\mu\alpha} R^\alpha{}_{\nu\rho\sigma}$. This has the following interesting properties:

i) It is anti-symmetric under the exchange $(\rho \leftrightarrow \sigma)$, so $R_{\mu\nu(\rho\sigma)} = 0$.

ii) It is also true that $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$.

iii) It is symmetric with respect to the four indexes, $R_{[\mu\nu\rho\sigma]} = 0$.

To prove i) and ii) it is useful to evaluate $R_{\mu\nu\rho\sigma}$ in a coordinate system where $\Gamma^k_{ij}|_p = 0$ and $\Gamma^k_{ij,l} = \frac{1}{2} g^{km}(\partial_l \partial_i g_{mj} + \partial_l \partial_j g_{mi} - \partial_l \partial_m g_{ij})$. Taking into account these properties, we conclude that the number of distinct components of the Riemann tensor is $n^2(n^2 - 1)/12$, which in four dimensions is 20.

[2] Watch out, in Wald's textbook [2] a different convention is used, namely $R^\mu{}_{\nu\rho\sigma}|_{\text{here}} = R_{\sigma\rho\nu}{}^\mu|_{\text{there}}$.

An interesting physical interpretation of the curvature tensor is that of *geodesic deviation.* Let U and ξ be two vector fields, such that U is geodesic and $\pounds_U \xi = 0$. In this situation, the "acceleration" of ξ along the integral curves of U is given by

$$\nabla_U \nabla_U \xi = R(U, \xi)(U)\,, \tag{2.21}$$

or, equivalently,

$$U^\mu U^\nu \nabla_\mu \nabla_\nu \xi^\sigma = R^\sigma{}_{\nu\mu\rho} U^\mu \xi^\rho U^\nu\,. \tag{2.22}$$

This observation is helpful to discuss tidal forces in black holes or understand the effect of gravitational waves on test particles.

From the Riemann tensor, the Ricci tensor is defined as $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$; and the Ricci scalar as its trace, $R = g^{\alpha\beta} R_{\alpha\beta}$.

2.5 Extrinsic curvature

If you were ask about your intuition of curvature, you would probably think of objects bent in particular ways. This is somehow related to the concept of *intrinsic curvature*, which is nothing but the curvature inherited by a submanifold N that is embedded in a higher dimensional manifold M. Let us make this precise.

Consider a $n-1$ dimensional hypersurface Σ embedded in a n dimensional space M. We assume that the points on Σ are characterised by $f(x^1, \ldots, x^n) = C =$ constant. Consequently,

$$\mathrm{d}f = \frac{\partial f}{\partial x^i} \mathrm{d}x^i \tag{2.23}$$

is a one-form normal to the hypersurface. We can get convinced of this by considering any curve $x^i(\lambda)$ contained in Σ, for which $f(x^i(\lambda)) = C$ and therefore $\mathrm{d}f(\mathrm{d}/\mathrm{d}\lambda) = \mathrm{d}f/\mathrm{d}\lambda = (\partial f/\partial x^i)(\mathrm{d}x^i/\mathrm{d}\lambda) = 0$.

Next, it is useful to define $|\mathrm{d}f|^2 := g^{\mu\nu}\partial_\mu f \partial_\nu f$. This is, in a sense, the norm of the one-form in Eq. (2.23). We say that[3] Σ is spacelike when $|\mathrm{d}f|^2 < 0$, as Σ and $\mathrm{d}f$ are perpendiculars in the sense explained before. Similarly, we say that Σ is timelike when $|\mathrm{d}f|^2 > 0$ and null or light-like if $|\mathrm{d}f|^2 = 0$. Whenever $|\mathrm{d}f|^2$ does not vanish, we can define the *unit-normal* to Σ

$$n = \frac{\mathrm{d}f}{\sqrt{|g^{\mu\nu}\partial_\mu f \partial_\nu f|}}\,. \tag{2.24}$$

Then $n^\mu n_\mu = 1$ if Σ is timelike and $n^\mu n_\mu = -1$ if it is spacelike. We use it do define the *induced metric* on Σ

$$h_{\mu\nu} = g_{\mu\nu} \pm n_\mu n_\nu\,, \tag{2.25}$$

where we would pick the $+$ $(-)$ sign for spacelike (timelike) hypersurfaces. The induced metric fulfils that $h_{\mu\nu}n^\mu = 0$ and that $V_\mu n^\nu = 0$ and $V_\mu h^{\mu\nu} = V^\nu$ for any vector V tangent to Σ. When written as a matric, h is also referred to as the *first fundamental form of* Σ.

From the induced metric, the *extrinsic curvature tensor* $K_{\mu\nu}$ is defined as

$$K_{\mu\nu} = h^\rho_\mu \nabla_\rho n_\nu\,. \tag{2.26}$$

[3] We consider only cases in which $|\mathrm{d}f|^2$ never changes sign.

It is also referred to as the *second fundamental form*. It is symmetric, and also fulfils that $K_{\mu\nu}n^\nu = 0$. Eventually, it will also be useful to consider its trace $K = h^{\mu\nu}K_{\mu\nu}$.

3 Spherically symmetric spaces

3.1 Spherical symmetry considerations

Definition 1. *A spacetime is said to be **spherically symmetric** if the subgroup of isometries contains* SO(3) *and the orbits generated by the isometries are two-spheres.*

This means that, if we pick any point of the spacetime and compute its orbit by the action of the group, we obtain a two sphere. Put even in a different way, each point in the manifold will be sitting in a two-dimensional surface, whose metric will be proportional to that of a unit two-sphere

$$\mathrm{d}s^2_{\mathrm{S}^2} = f(\rho,\tau)^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) = f(\rho,\tau)^2 \mathrm{d}\Omega_2. \tag{3.1}$$

Note that the function $f(\rho,\tau)$ is related to the area of the sphere determined by setting the other two coordinates to a constant, $\tau =$ constant and $\rho =$ constant. Specifically, the area is $A(\rho,t) = 4\pi f(\rho,\tau)^2$.

It is often useful to perform a change of coordinates $(\rho,\tau) \to (r,\tau)$ defined by $r = f(\rho,\tau)$. Because in terms of r the area reads $A(\rho,t) = A(r) = 4\pi^2 r^2$, this coordinate is sometimes referred to as *area coordinate* or *curvature coordinate*. Note that since we are working in a curved spacetime, generically it will not coincide to the distance to the "center" of the sphere. Actually, in some cases we will not be able even to talk about such center, as we will see. Nevertheless, we will still sometimes call it *radial coordinate*.

Because we want the full four dimensional metric to be spherically symmetric, it has to preserve the isometries in Eq. (3.1). In particular, it will be written as

$$\mathrm{d}s^2 = g_{\tau\tau}(r,\tau)\mathrm{d}\tau^2 + 2g_{\tau r}(r,\tau)\mathrm{d}\tau\mathrm{d}r + g_{rr}(r,\tau)\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2. \tag{3.2}$$

The reason why the terms $\mathrm{d}t\mathrm{d}\alpha_i$ and $\mathrm{d}r\mathrm{d}\alpha_i$ (with $\alpha_i = \varphi,\, \theta$) do not appear is that they would break rotational symmetry (for instance, the metric would not be invariant under $\alpha_i \to -\alpha_i$, while $\mathrm{d}\Omega_2$ is). Furthermore, we can always make a coordinate transformation $(r,\tau) \to (r,t)$ in such a way that the metric takes the form

$$\mathrm{d}s^2 = -e^{2\phi(t,r)}\mathrm{d}t^2 + e^{2\Lambda(t,r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2. \tag{3.3}$$

Before continuing, it is worth talking about two concepts that are often discussed (and confused) in the literature: stacionarity and staticity.

Definition 2. *We say that a spacetime is **stationary** if it possesses a one-parameter group of isometries, ϕ_t, whose orbits are timelike.*

Physically, this mean that we are dealing with a "time translational symmetric" spacetime. That is to say, it possesses a Killing vector ξ^μ whose orbits are timelike curves. In particular, note that the metric in Eq. (3.3) will be stationary if $\phi(t,r) = \phi(r)$ and $\Lambda(t,r) = \Lambda(r)$, since in that case $\partial/\partial t$ becomes a Killing vector. Moreover,

Definition 3. *We say that an stationary spacetime is also **static** if there is a spacelike hypersurface which is orthogonal to all the orbits of the isometry at every point.*

As we mentioned, it is possible to eliminate the $g_{\tau r}$ component in Eq. (3.2) by performing a change of coordinates. thus, we conclude that any stationary, spherically symmetric spacetime is also static.

3.2 Birkhoff's theorem and the Schwarschild solution

Let's now compute the Levi-Cività connection of the metric Eq. (3.3):

$$\begin{aligned}
&\Gamma^t_{tt} = \dot{\phi}\,, \quad \Gamma^t_{tr} = \phi'\,, \quad \Gamma^t_{rr} = \dot{\Lambda} e^{2\Lambda-2\phi}\,,\\
&\Gamma^r_{tt} = \phi' e^{2\phi-2\Lambda}\,, \quad \Gamma^r_{tr} = \dot{\Lambda}\,, \quad \Gamma^r_{rr} = \Lambda'\,, \quad \Gamma^r_{\theta\theta} = -re^{-2\Lambda}\,, \quad \Gamma^r_{\varphi\varphi} = -re^{-2\Lambda}\sin^2\theta\,,\\
&\Gamma^\theta_{r\theta} = 1/r\,, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta\,, \quad \Gamma^\varphi_{r\varphi} = 1/r\,, \quad \Gamma^\varphi_{\theta\varphi} = \cot\theta\,;
\end{aligned} \tag{3.4}$$

while the rest of components that are not obtained by symmetrisation of the lower indexes of the previous, are zero. Once the Levi-Cività connection is known, we can compute the Riemann tensor and extract from it the Ricci tensor, which appears in Einstein's equations:

$$\begin{aligned}
R_{tt} &= e^{2\phi-2\Lambda}\left(\phi'' + (\phi')^2 - \phi'\Lambda' + \frac{2\phi'}{r}\right) - \ddot{\Lambda} - \dot{\Lambda}^2 + \dot{\Lambda}\dot{\phi}\,,\\
R_{rr} &= -\phi'' - (\phi')^2 + \phi'\Lambda' + \frac{2\Lambda'}{r} + e^{2\Lambda-2\phi}\left(\ddot{\Lambda} + \dot{\Lambda}^2 - \dot{\Lambda}\dot{\phi}\right)\,,\\
R_{\theta\theta} &= 1 - e^{-2\Lambda}\left(1 + r(\phi' - \Lambda')\right)\,,\\
R_{\varphi\varphi} &= \sin^2\theta R_{\theta\theta}\,, \qquad R_{rt} = \frac{2\dot{\Lambda}}{r}\,.
\end{aligned} \tag{3.5}$$

All the non-diagonal components but R_{rt}, R_{tr} are zero. We can now prove the following very important theorem:

Theorem 1. *(Birkhoff's theorem) Any solution of Einstein's empty space equations which is spherically symmetric, is locally equivalent to the Schwarzschild solution, namely,*

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 - \frac{2M}{r}\right)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)\,. \tag{3.6}$$

In particular, the metric components do not depend on t, so $\partial/\partial t$ is a Killing vector.

Proof. In order to proof this theorem, we just have to solve Einstein's equations Eq. (1.1) in the absence of matter, $T_{\mu\nu} = 0$. Taking the trace of these equations, we immediately see that in vacuum the Ricci scalar has to vanish, $R = 0$. Therefore, Einstein's equations in the absence of matter reduce to

$$R_{\mu\nu} = 0. \tag{3.7}$$

The most general spherically symmetric metric was already written down in Eq. (3.3), and the components of the Ricci tensor are given in Eq. (3.5). If we first impose $R_{tr} = 0$, we find that $\dot{\Lambda} = 0$, which implies that

$$\Lambda(r,t) = \Lambda(r) + C_1\,. \tag{3.8}$$

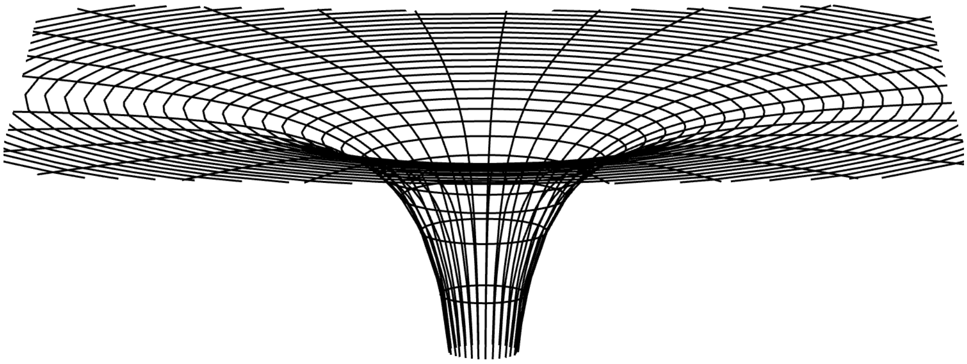

Figure 2: Illustration of the Schwarschild metric, Eq. (3.6).

On the other hand, substituting Eq. (3.8) into the equation $R_{\theta\theta} = 0$, we find that ϕ' is just a function of r. In particular,

$$\phi(r,t) = \tilde{\phi}(r) + C_2(t)\,, \tag{3.9}$$

Because it is always possible to find a change of coordinates such that $\mathrm{d}t' = e^{C_2(t)}\mathrm{d}t$, in which case $e^{2\phi(r,t)}\mathrm{d}t^2 = e^{2\tilde{\phi}(r)}\mathrm{d}t'^2$, we conclude that in vacuum all dependence in t disappears from the metric components. Therefore, after renaming the coordinates and the functions, we can write the metric as

$$\mathrm{d}s^2 = -e^{2\phi(r)}\mathrm{d}t^2 + e^{2\Lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2. \tag{3.10}$$

To finalise the prove and find Eq. (3.6), we still have to solve $R_{tt} = 0$ and $R_{rr} = 0$. The combination $e^{2\phi-2\Lambda}R_{rr} + R_{tt} = 0$ leads to the relation

$$(\Lambda' + \phi')\frac{1}{r} = 0\,, \tag{3.11}$$

from which we deduce that $\phi' = -\Lambda'$. In particular, $\phi(r) = -\Lambda(r) + A$, and we can set $A = 0$ since it can always be reabsorbed by a convenient rescaling of the time coordinate. Finally, taking this into consideration in equation $R_{\theta\theta} = 0$, we find that

$$1 - 2r\Lambda' = e^{2\Lambda}\,, \tag{3.12}$$

whose solution is $e^{2\Lambda} = (1 + B/r)^{-1}$. So far, we see that Eq. (3.10) now reads

$$\mathrm{d}s^2 = -\left(1 + \frac{B}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 + \frac{B}{r}\right)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\,, \tag{3.13}$$

which is nothing but Eq. (3.6) by setting $B = -2M$. □

The reason why we normally choose this value for this last integration constant is the following. Note that for large values of r this metric approaches that of Minkowski spacetime. Spacetimes with this property are said to be **assymptotically flat**. In this limit, we should recover Newtonian gravity, as we expect (3.13) to describe the effect of gravity in the non-relativistic limit. Looking at the tt components,

$$g_{00} = -\left(1 + \frac{B_2}{r}\right) = \eta_{tt} + h_{tt} + \mathcal{O}(h^2) = -1 - \frac{2\Phi_N}{c^2} + \mathcal{O}(h^2)\,, \tag{3.14}$$

where we have used that $h_{tt} = -2\Phi_N/c^2$ with $\Phi_N = GM/r$ the Newtonian potential, as already discussed when finding Einstein's equations. From this requirement we get $B_2 = -2GM/c^2$, and thus $2M$ in geometrised units.

Let us make one last comment regarding the radial coordinate r in Schwarschild geometry. We said previously that this should not be thought of as the distance to the center on any sphere. Still, the area of the sphere at which an event (t, r, θ, φ) is sitting is $A(r) = 4\pi r^2$ by construction. However, say that we want to move from the sphere with $r = r_1$ to the sphere with $r = r_2$. The distance that we will have to travel is

$$L = \int_{r_1}^{r_2} \frac{\mathrm{d}r}{\sqrt{1 - \frac{2M}{r}}} = r\sqrt{1 - \frac{2M}{r}} + 2M\mathrm{arctanh}\sqrt{1 - \frac{2M}{r}}\Bigg|_{r_1}^{r_2} > r_2 - r_1\,. \tag{3.15}$$

This distance is longer that the one that should be travelled if the space was flat, as illustrated in Fig. 2. The picture represents the ratio between the value of the distance traveled starting at $r = 2M$ in this curved geometry over the corresponding quantity in flat space.

Finally, note that the metric Eq. (3.6) is ill-behaved at $r = 2M$, where the g_{tt} component of the metric vanishes and the g_{rr} component blows up. This particular value of the radial coordinate is often referred to as the *Schwarschild radius*, and denoted by $r_s = 2M$. The area of the corresponding sphere sitting at r_s grows linearly with the mass M of the object sourcing the geometry. However, it is important to notice that, for most of the astrophysical objects, r_s is normally way smaller that the characteristic size of the object. For example, for the Sun $r_s \simeq 1.47\,\mathrm{km}$, while its radius is $R_\odot = 6.96 \times 10^7\,\mathrm{km} \gg r_s$. On the other hand, the mass of typical neutron stars is around 1.5 solar masses, so their Schwarschild radius $r_s \simeq 2.2\,\mathrm{km}$ starts to be comparable to their radius $\sim 11\,\mathrm{km}$. Note, however, that because inside these objects $T_{\mu\nu} \neq 0$, the metric in Eq. (3.6) only describes their exterior part, so we need not worry about what happens at $r = 2M$ to describe them.

Eventually, is we manage to gather enough matter together in the same region of space, nothing will prevent its gravitational collapse, and the surface $r = r_s$ will be realised. In the next Section, we will argue that this is the case by solving Einstein's equation in presence of matter.

3.3 Relativistic Stars

In this Section we imagine that the spacetime in Eq. (3.6) is sourced by some spherical distribution of mass siting at "the center" of the spacetime. We can think of this as a convenient modelling of a star. Of course, this will be a simplified model, among other reasons because we expect stars to have a certain rotation, which we do not include since it would break spherical symmetry. A second simplification that we take is that the fluid the star is made of is a *perfect fluid*, whose energy-momentum tensor takes the form

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \,, \tag{3.16}$$

where U^μ is the quadri-velocity of the points inside the star. Because these points are standing still in the equilibrium state, they are such that r, θ, φ = constant. Thus, the unique non–trivial component $U^r = e^\phi$ is determined by requiring $U_\mu U^\mu = -1$. Consequently, the components of the energy-momentum tensor read

$$T_{tt} = \rho e^{2\phi}, \quad T_{rr} = p e^{2\Lambda}, \quad T_{\theta\theta} = p r^2, \quad T_{\varphi\varphi} = \sin^2\theta\, T_{\theta\theta} \,. \tag{3.17}$$

In contrast to the vacuum case, stationarity and staticity of the solution is not obtained for free, but needs to be imposed as an assumption. In particular, we set all the time derivatives in Ricci tensor (Eq. 3.5) and the Christoffel symbols (Eq. (3.4)) to zero. In this situation, the Ricci scalar becomes

$$R = -2e^{-2\Lambda}\left(\phi'' + (\phi')^2 - \phi'\Lambda' + \frac{2}{r}(\phi' - \Lambda') + \frac{1}{r^2}\right) + \frac{2}{r^2} \,. \tag{3.18}$$

We can now solve Einsteins equations´, which in this case it turn out useful to be written in the form $R^\mu{}_\nu - \frac{1}{2}Rg^\mu{}_\nu = 8\pi T^\mu{}_\nu$. From the $R^t{}_t - \frac{1}{2}Rg^t{}_t = 8\pi T^t{}_t$ we get

$$-8\pi\rho = e^{-2\Lambda}\left(\frac{2\Lambda'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} \,. \tag{3.19}$$

This equation tells us that

$$e^{-2\Lambda} = 1 - \frac{2m(r)}{r} \,, \qquad \text{with} \quad m(r) := m(0) + \int_0^r 4\pi\rho r^2 \mathrm{d}r \,. \tag{3.20}$$

Note that we have left explicit the boundary condition $m(0)$, which is determined by demanding regularity at the center of the star. Indeed, if we want space to be locally flat at $r = 0$, we have to demand that $m(0) = 0$, so that $e^{2\Lambda} \to 1$ as $r \to 0$.

Let us now analise the equation $R^r{}_r - \frac{1}{2}Rg^r{}_r = 8\pi T^r{}_r$, we find

$$8\pi p = e^{-2\Lambda}\left(\frac{2\phi'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \,. \tag{3.21}$$

We can substitute the solution for Λ that we found in Eq. (3.20) into this equation and find the solution for ϕ, which reads

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{m(r) + 4\pi p r^3}{r(r - 2m(r))} \,. \tag{3.22}$$

We have now determined the functions of the metric, but are still lacking an equation to know how matter is distributed inside the star. This comes from the conservation of the energy momentum tensor $T^{\mu\nu}$. Indeed, the Bianchi identity for the Einstein tensor implies that $\nabla_\mu T^{\mu\nu} = 0$. In the present scenari it reduces to

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -(p+\rho)\frac{\mathrm{d}\phi}{\mathrm{d}r}\,. \tag{3.23}$$

In this case, the boundary condition is such that the pressre has some particular value at the center of the star, $p(0) = p_0$.

We have arrived to the Tolman–Oppenheimer–Volkoff (TOV) equations,

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho+p)(m+4\pi r^3 p)}{r(r-2m)}\,, \qquad \frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\rho r^2\,; \tag{3.24}$$

where the equation of state determines $\rho(p)$. The boundary conditions at the center of the star are $m(0) = 0$ and $p(0) = p_0$. Note that the first equation in Eqs. (3.24) tells us that the pressure decreases as r increases. Eventually, it will vanish at some $r = R$, which we declare is the radius of the star. There, the metric components should be matched to the Schwarschild metric Eq. (3.6), as this is the unique spacetime metric that describes a spherically symmetric vacuum solution. In particular, $m(R) = M$.

The TOV equations can be solved for the (unrealistic) case of a constant density star. In this case one obtaines that the ratio between the radius and the mass of the star is bounded from above,

$$\frac{R}{M} \leq \frac{9}{4}\,. \tag{3.25}$$

In the limit case $R = 9M/4$ the pressure at the center of the star becomes infinite, which means that the star cannot sustain more mass. Actually, Hans Adolf Buchdahl proved that the bound in Eq. (3.25) is true for any equation of state (*Buchdahl theorem*). This result suggest that, whatever kind of matter we consider, if we manage to gather enough of it within a region of space, its equation of state will be unable to prevent gravitational collapse.

3.4 Trajectories in Schwarschild spacetime

The spacetime metric that we found in Eq. (3.6) is supposed to describe how spacetime is deformed around spherical objects of mass M. In this spacetime, we expect lighter objects to move following geodesics, without deforming the metric very much (probe approximation).

To examine the predictions that this model leads to, we can follow a similar procedure as the one followed when studying gravitational systems in classical mechanics. In that case, many observational consequences such as Kepler's law could be derived from Newton's law of gravity. Let us then study what kind of trajectories do bodies follow when moving in the Schwarschild metric. So far, we know that they must follow geodesics.

The two kind of objects that we want to study are particles (planets, stars,...) whose mass m is small compared to M; and photons (light rays), which are massless. We will denote the momentum of particles by P, $P_\alpha P^\alpha = -m^2$; and the momentum of photons by K, $K_\alpha K^\alpha = 0$. Both P and K are geodesic and, for that reason, their modulus is constant along the trajectories.

First of all we can ask whether there is any conserved quantity in our system. Recall that for every Killing vector, there will be a conserved quantity along the trajectory of the object, as we saw in Sec. 2.3. In particular, we have two Killing vectors, $\xi_{(1)} = \partial/\partial t$ and $\xi_{(2)} = \partial/\partial\varphi$, which lead to two conserved quantities (energy and angular momentum respectively). The conserved quantities for particles will then be

$$g_{\alpha\beta}\xi^{\alpha}_{(1)}P^{\beta} = P_t =: -\hat{E}m\,, \qquad g_{\alpha\beta}\xi^{\alpha}_{(2)}P^{\beta} = P_{\varphi} =: -\hat{L}m\,; \tag{3.26}$$

whereas for photons,

$$g_{\alpha\beta}\xi^{\alpha}_{(1)}K^{\beta} = K_t =: -E\,, \qquad g_{\alpha\beta}\xi^{\alpha}_{(2)}K^{\beta} = K_{\varphi} =: L\,. \tag{3.27}$$

If an object is originally on the equatorial plane, $\theta = \pi/2$, it will not leave it due to the uniqueness of solutions of ordinary differential equations, since our system is invariant under $\theta \to \pi - \theta$. If the object was not originally at $\theta = \pi/2$, we can always rotate the coordinates $(\theta, \varphi) \to (\tilde{\theta}, \tilde{\varphi})$ so that the new equatorial plane $\tilde{\theta}$ coincides with the plane spanned by the momentum and $\nabla_i r$. Consequently, we can assume without loss of generality that all the trajectory is contained in the equatorial plane, and thus we set $\theta = \pi/2$.

For particles we have

$$P_{\mu}P^{\mu} = -m^2 = g^{tt}(P_t)^2 + g^{rr}(P_r)^2 + g^{\varphi\varphi}(P_{\varphi})^2 = m^2\left[g^{tt}\hat{E}^2 + g^{\varphi\varphi}\hat{L}^2 + g^{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2\right]\,, \tag{3.28}$$

and then the trajectory is given by

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \hat{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right) = \hat{E}^2 - \hat{V}^2(r)\,. \tag{3.29}$$

Analogously, for photons we get a very similar expression,

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} = E^2 - V^2(r)\,. \tag{3.30}$$

Interestingly, both Eqs. (3.29) and (3.30) took the same form, $(\mathrm{d}r/\mathrm{d}\mu)^2 = \tilde{E}^2 - \tilde{V}^2(r)$, where $\mu = \tau\,, \lambda$ stands for the corresponding affine parameter. This is interesting because, deriving on both sides by the affine parameter and using the chain rule we obtain

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\mu^2} = -\frac{1}{2}\frac{\mathrm{d}\tilde{V}^2}{\mathrm{d}r}\,. \tag{3.31}$$

We can phrase this last expression as "the acceleration is the derivative of the potential"[4]. Let us see what we can learn from this equation regarding the motion of objects, separately for particles and photons.

■ **Particles.** For particles, the effective potential reads

[4] This is only a convenient way to phrase it, even though is not very precise, as these objects are free falling and therefore their acceleration is zero. By "acceleration", we meant the second derivative of the radial coordinate with respect to the affine parameter, which is not the actual acceleration since the objects are moving also in the angular direction).

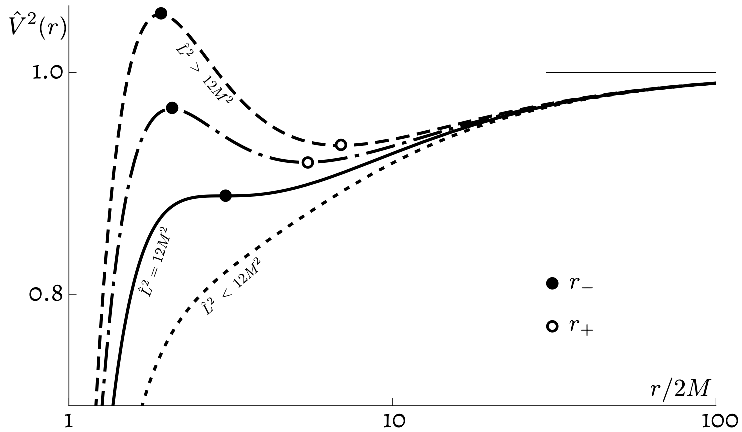

Figure 3: Different effective potentials for particles obtained depending on the value of $\hat{L}$. Note that for all cases the potential approaches 1 at infinity. The position of the possible unstable and stable circular orbits are represented by the filled and empty circles respectively.

$$\hat{V}^2(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right) . \tag{3.32}$$

Note that $\hat{V}^2(2M) = 0$. At the same time, $\mathrm{d}\hat{V}/\mathrm{d}r$ vanishes at two different points,

$$r_\pm = \frac{\hat{L}^2}{2M}\left[1 \pm \sqrt{1 - \frac{12M^2}{\hat{L}^2}}\right] , \tag{3.33}$$

provided $\hat{L}^2 > 12M^2$. There, if $\hat{V}^2(r_\pm) = \hat{E}^2$, circular orbits are possible. However, note that the orbit at r_- is unstable, since it corresponds to a maximum of $\hat{\hat{V}}^2(r)$. In contrast, at r_+ there is an stable circular orbit.

Additionally, note that, when $\hat{V}^2(r_+) < \hat{E}^2 < 1$, orbits around r_+ are possible (even though they will not be closed, as we will see later). Particles whose energy is $1 < \hat{E}^2 < \hat{V}^2(r_-)$ will approach our massive object and scape again to infinity. Finally, a particle with $\hat{E} > \hat{V}^2(r_-)$ will keep falling unavoidably.

On the other hand, when $\hat{L}^2 < 12M^2$ the argument of the square root in Eq. (3.33) is negative and therefore $\mathrm{d}\hat{V}/\mathrm{d}r$ does not vanish for any real value of r. In this case, no matter what the value of $\hat{E}$ is, particles keep falling.

Finally, for the fine-tunned case $\hat{L}^2 = 12M^2$, the root of $\mathrm{d}V^2(r)/\mathrm{d}r$ becomes double since $r_+ = r_- = 6M = 3r_s$ and $V^2(r)$ develops an inflexion point. For this reason, any circular stable orbit will have radius larger than $3r_s$.

■ **Photons.** In this case, the effective potential reads

$$\hat{V}^2(r) = \left(1 - \frac{2M}{r}\right)\frac{\hat{L}^2}{r^2} . \tag{3.34}$$

The analysis for photons is simpler, since the qualitative behaviour does not depend on the ratio between L and M. Indeed, $\mathrm{d}\hat{V}^2/\mathrm{d}r$ always vanishes at $r = 3r_s/2$, where

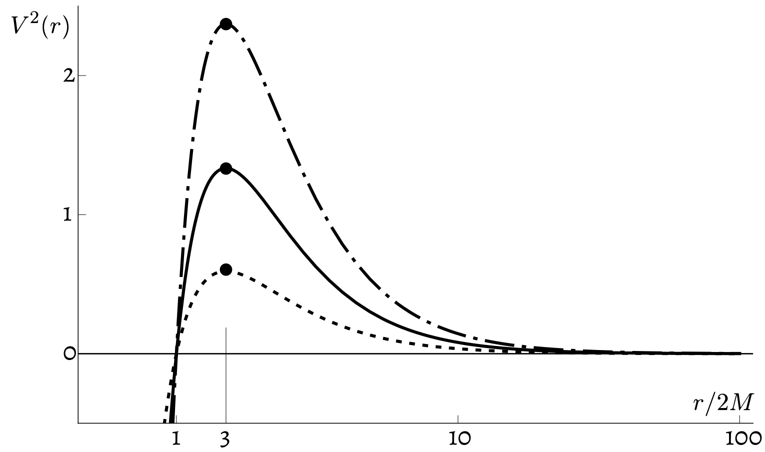

Figure 4: Different effective potentials for photons, for different choices of L. Unstable circular orbits are represented by filled circles, and are such that $r = 6M$.

$V^2(r)$ develops a maximum. Note this means that at this position there can be a circular orbit of photons if $E^2 = \hat{V}^2(3r_s/2)$, though it is unstable. If E^2 is smaller than that value, photons will escape back to infinity; whereas they will keep falling if their energy is bigger.

3.5 Mercury's perihelion

Kepler's first law states that planets describe ellipses around the Sun. Solving the two body problem in Newtonian gravity, this is proven to be indeed the case, as we will review in this section. However, when effects such as the presence of other bodies (for example, the other planets) or non-sphericity of the Sun are taken into consideration, the orbits are not perfect ellipes anymore. Rather, the position of their perihelion changes. When these effects are taken into account, Newtonian gravity predicts that Mercury's perihelion should advance 5557″ per century. However, while Einstein was developing his theory, the observed value for the advance of Mercury's perihelion was 5600″ per century, which meant that there was a 43″ per century discrepancy that Newtonian gravity was not able to explain (see Fig. 5). Can General Relativity account for this discrepancy?

To answer this question, we wish to study orbits of particles moving in the spacetime Eq. (3.6), and for that we need to determine how r depends on the angle φ; that is to say $r(\varphi)$. Taking into consideration that $\mathrm{d}\varphi/\mathrm{d}\tau = m^{-1}P^{\varphi} = \hat{L}/r^2$, we can rewrite Eq. (3.29) as

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \frac{r^4}{\hat{L}^2}\left[\hat{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right)\right] . \tag{3.35}$$

It is now useful to change the radial coordinate to $u = 1/r$. Then, $\mathrm{d}u/\mathrm{d}\varphi = -r^{-2}\mathrm{d}r/\mathrm{d}\varphi = -u^2\mathrm{d}r/\mathrm{d}\varphi$ and (3.35) becomes

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = \frac{\hat{E}^2 - 1}{\hat{L}^2} + \frac{2M}{\hat{L}^2}u - u^2 + 2Mu^3 . \tag{3.36}$$

To find the solution to our problem, it will turn useful to refresh how elliptical orbits are found in the Newtonian case.

Cause	m^{-1}	Motion of perihelion: Mercury	Motion of perihelion: Earth
Mercury	6 000 000 ±1 000 000	0″.025±0″.00	−13″.75±2″.3
Venus	408 000 ± 1 000	277.856±0.68	345.49±0.8
Earth	329 390 ± 300	90.038±0.08	
Mars	3 088 000 ± 3 000	2.536±0.00	97.69±0.1
Jupiter	1 047.39± 0.03	153.584±0.00	696.85±0.0
Saturn	3 499 ± 4	7.302±0.01	18.74±0.0
Uranus	22 800 ± 300	0.141±0.00	0.57±0.0
Neptune	19 500 ± 300	0.042±0.00	0.18±0.0
Solar oblateness		0.010±0.02	0.00±0.0
Moon			7.68±0.0
General precession (Julian century, 1850)		5025.645±0.50	5025.65±0.5
Sum		5557.18 ±0.85	6179.1 ±2.5
Observed motion		5599.74 ±0.41	6183.7 ±1.1
Difference		42.56 ±0.94	4.6 ±2.7
Relativity effect		43.03 ±0.03	3.8 ±0.0

Figure 5: Different contribution to the motion of the perihelia of Mercury and the Earth. As can be seen, the General Relativity accounts for the discrepancy between the observation and the theoretical value using Newtonian gravity. Table taken from [3], see [4] for an update from 2017.

■ **Newtonian case.** As we shall argue, in the non-relativistic limit we have an expression similar to Eq. (3.36). The difference is just that the last term is missing. Indeed, recall that in the Newtonian case we also had two conserved quantities:

$$\begin{aligned} E_N &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{mM}{r}\,, \\ L &= mr^2\dot{\varphi}\,. \end{aligned} \tag{3.37}$$

The second equality can be written as $\dot{\varphi} = \hat{L}/r^2$, which we can in turn substitute in the first one. After performing the change of variables $r = 1/u$, it becomes

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = \frac{2\hat{E}_N}{\hat{L}^2} + \frac{2M}{\hat{L}}u - u^2\,, \tag{3.38}$$

which is nothing but Eq. (3.36) after neglecting the last term. We can relate the Newtonian energy E_N to the relativistic one E at infinity ($r \to \infty$), where the metric is asymptotically flat. There, as in Special Relativity

$$P^t = E = m\gamma(v_\infty) = m + m\frac{v_\infty^2}{2} + \mathcal{O}(v_\infty^4) = m + E_N + \mathcal{O}(v_\infty^4) \tag{3.39}$$

Consequently $\hat{E}^2 = 1 + 2E_N + \mathcal{O}(v^4)$. Therefore, after completing the square, we can rewrite Eq. (3.38) as

$$\left[\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(u - \frac{M}{\hat{L}^2}\right)\right]^2 + \left(u - \frac{M}{\hat{L}^2}\right)^2 = \frac{1}{\hat{L}^2}\left(\hat{E}^2 - 1 + \frac{M^2}{\hat{L}^2}\right)\,. \tag{3.40}$$

This is of the form $f'(x)^2 + f(x)^2 = f_0^2$, whose solution is $f(x) = f_0 \cos(x - x_0)$. We thus conclude that in Newtonian gravity, as we knew, orbits are ellipses described by

$$\frac{1}{r} = \frac{M}{\hat{L}^2} + \frac{1}{\hat{L}} \left(\hat{E}^2 - 1 + \frac{M^2}{\hat{L}^2} \right)^{\frac{1}{2}} \cos(\varphi - \varphi_0) \,. \tag{3.41}$$

Note that, when the quantity inside the parenthesis vanishes, we obtain circular orbits. We can interpret the quantity

$$y = u - \frac{M}{\hat{L}^2} = \frac{1}{r} - \frac{M}{\hat{L}^2} \tag{3.42}$$

as the deviation from circularity.

■ **Relativistic correction.** In order to compute the relativistic correction to the Newtonian orbit, it is useful to express our original Eq. (3.36) in terms of the deviation from circularity, defined in Eq. (3.42). In that case we get

$$\left(\frac{\mathrm{d}y}{\mathrm{d}\varphi} \right)^2 = \frac{1}{\hat{L}^2} \left(\hat{E}^2 - 1 + \frac{M^2}{\hat{L}^2} \right) + \frac{2M^4}{\hat{L}^6} + \frac{6M^3}{\hat{L}^4} y + \left(\frac{6M^2}{\hat{L}^2} - 1 \right) y^2 + 2My^3 \,. \tag{3.43}$$

In this expression we have still not taken any approximation. If the departure from circularity is mild, the last term can be neglected[5]. Without that term, we can complete the square and rewrite the expression as

$$\left(\frac{\mathrm{d}}{\mathrm{d}\varphi} (y - y_0) \right)^2 + k(y - y_0)^2 = \frac{1}{\hat{L}^2} \left(\hat{E}^2 - 1 + \frac{M^2}{\hat{L}^2} \right) + \frac{2M^4}{\hat{L}^6} + k^2 y_0^2 \tag{3.44}$$

where

$$k = \left(1 - \frac{6M^2}{\hat{L}^2} \right)^{\frac{1}{2}} \,, \qquad y_0 = \frac{3M^3}{\hat{L}^4 k^2} \,. \tag{3.45}$$

The solution then takes again the form

$$y(\varphi) = y_0 + A \cos(k\varphi - \varphi_0) \,, \qquad A = k^{-1} \left(\frac{\hat{E}^2 - 1 + M^2/\hat{L}^2}{\hat{L}^2} + \frac{2M^4}{\hat{L}^6} + k^2 y_0^2 \right)^{\frac{1}{2}} \,. \tag{3.46}$$

Interestingly, the appearance of k in the argument of the cosine changes the period of the function to $\Delta\varphi = 2\pi/k$. The perihelion, appearing every time the cosine reaches

[5] Mercury's eccentricity may is $e \approx 0.2$ and may not seem negligible. Let us examine in this footnote the applicability of this approximation. For the particular case of Mercury and the Sun, the order of magnitude of the angular momentum can be approximated from Mercury's semi-major axis and eccentricity (see Eq. (3.48) in the main text below) to be such that $\hat{L}^2 \approx 8.2 \times 10^7\,\mathrm{km}^2$. In particular, $\hat{L} \approx 6 \times 10^3 M$.

On the other hand, Mercury's separation from the Sun oscillates between $r_{\max} \approx 70 \times 10^6\,\mathrm{km}$ and $r_{\min} \approx 46 \times 10^6\,\mathrm{km}$, which implies that $y_{\max} \approx 4.5 \times 10^{-9}\,\mathrm{km}^{-1} \approx 6.7 \times 10^{-9} M^{-1}$. At the end of the day, this means that the term with y^2 in Eq. (3.44) is of order $4.5 \times 10^{-17} M^{-2}$, while the y^3 is about $6.1 \times 10^{-25} M^{-2}$, that is, 10^6 times smaller. Actually, the term $6M^3 y/\hat{L}^4 \sim 2.7 \times 10^{-23} M^{-2}$ is similarly small and could also be neglected.

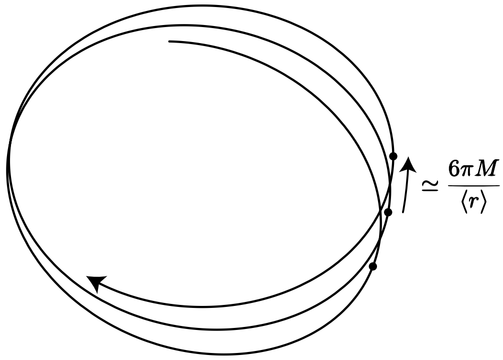

Figure 6: Representation of the shift in Mercury's perihelion. Actually, dots represent the *aphelion*, which is the point of maximum distance from the Sun, whose shift is of course the same as that of the perihelion.

its minimum, is not found at the same value of the angular variable, but it is shifted by

$$\Delta\varphi - 2\pi = 2\pi\left(\frac{1-k}{k}\right) \simeq \frac{6\pi M^2}{\hat{L}^2} \,. \tag{3.47}$$

The last approximation is considering $M/\hat{L}$ to be small. For a circular orbit, the reduced angular momentum $\hat{L}^2 = r^2v^2 = 4\pi^2 r^4/T^2 = Mr$, where in the last equality we used Kepler's third law. For an elliptical orbit this gets modified to

$$\hat{L}^2 = M(1-e^2)a \,, \tag{3.48}$$

where e is its eccentricity and a the value of the semi-major axis. Substituting it into Eq. (3.47), together with the mass of the Sun $M = 1.48$ km and Mercury's semi-major axis $a = 5.79 \times 10^{-7}$ km and excentricity $e = 0.206$, we obtain that $\Delta\varphi - 2\pi = 5.03 \times 10^{-7}$ rad per orbit. knowing that Mercury's orbital period is $T = 0.241$ years, we obtain $\Delta\varphi - 2\pi = 43.1''$ per century.

3.5.1 Deflection of light

In the previous section we considered a relativistic effect suffered by the orbit of a planet. Now, we study another effect that it is also measurable in the Solar system: the deflection of light by the Sun.

Photons are not affected by the gravitational field in Newtonian gravity[6]. However, we have already seen that their geodesics will move according to Eq. (3.30) in a curved background. We want to see the effect that this may have when the mass M sourcing the gravitational field is small (compared to the distance at which the light rays are passing

[6] Actually, it is also true that light do deflect in Newtonian gravity if photons had a small mass, and no-matter how small that mass would be they would always deflect by the same angle (independent of the mass), as they follow the hyperbolas determined by Eq. (3.41), as showed by Johann Georg von Soldner in 1804 [5]. The value obtained is half the result that we will get from Einstein's equations.

by). Recalling that $L = K_\varphi = r^2 \mathrm{d}r/\mathrm{d}\varphi$, Eq. (3.30) becomes

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \left(E^2 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}\right)\frac{r^4}{L^2}\,, \tag{3.49}$$

which in the coordinate $u = 1/r$ reads

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = -u^2\left(1 - 2Mu\right) + \frac{1}{b^2}\,. \tag{3.50}$$

Note that we find useful in this case to define the *impact parameter* $b = L/E$. The interpretation of this parameter is simple. In the limit $M \to 0$, Eq. (3.51) becomes

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2 = \frac{1}{b^2}\,, \tag{3.51}$$

whose solution is $1/r = u_0 = b^{-1}\sin(\varphi - \varphi_i)$, which is a straight line. The sub-index "0" stands for the fact that we have set M to zero. Moreover, we can set $\varphi_i = 0$, so that $r \to 0$ for $\varphi = 0$ and π. This solution is also a good approximation of the solution to the full Eq. (3.51) sufficiently far from the source, $2Mu = 2M/r \ll 1$.

We can now solve the equation perturbatively around $M = 0$. For that, in order to know how the solution u_0 is modified in the presence of a small mass M, we search for a solution of the form

$$u(\varphi) = u_0(\varphi) + M\delta u(\varphi) + \mathcal{O}(M^2)\,. \tag{3.52}$$

Before trying to solve it, it turns useful to differentiate Eq. (3.51) with respect to $\mathrm{d}/\mathrm{d}\varphi$. By doing so we obtain

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = 3Mu^2\,. \tag{3.53}$$

Now, substituting Eq. (3.52) into Eq. (3.53) we get

$$0 = \left(u_0'' + u_0\right) + \left(-3u_0^2 + \delta u + \delta u''\right)M + \mathcal{O}(M^2)\,. \tag{3.54}$$

The first term is of course solved by the unperturbed solution, u_0. Then the solution for δu turns out to be

$$\delta u = C_1\cos\varphi + C_2\sin\varphi + \frac{1}{2b^2}\left(3 + \cos(2\varphi)\right) = \frac{1}{b^2}\left(1 + \cos\varphi\right)^2\,. \tag{3.55}$$

In the last equality, we have been forced to set $C_2 = 0$ in such a way that our solution also solves Eq. (3.51), and $C_1 = -2/b^2$ so that at $\varphi = \pi$ (that is, initially) the perturbation vanishes, $\delta u(\pi) = 0$. In the end, the solution takes the form

$$u(\varphi) = \frac{1}{b}\sin\varphi_\infty + \frac{M}{b^2}\left(1 + \cos\varphi\right)^2\,. \tag{3.56}$$

The light ray is assymptotically far when $r \to \infty$, which means $u(\varphi) = 0$. This happens at $\varphi = \pi$ by construction, since we chose our "initial" boundary condition in this way. However there is another "final" angle for which $u(\varphi_\infty) = 0$ is realised. Assuming

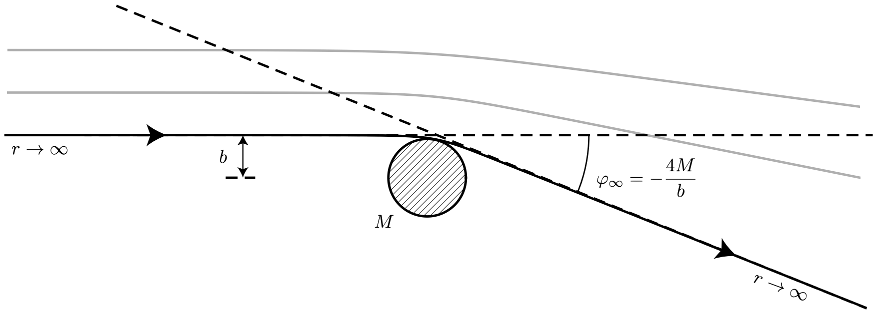

Figure 7: Deflection of light.

this angle will be small, we approximate $\sin\varphi_\infty \simeq \varphi_\infty$ and $\cos\varphi_\infty \simeq 1$ in Eq. (3.56) and obtain that the deflected angle is

$$\varphi_\infty = -\frac{4M}{b}\,, \tag{3.57}$$

which is indeed small whenever $M/b \ll 1$.

Let's pause for a moment and analyse this result. The following discussion is better understood by looking at Fig. 7. If there is no object deforming spacetime at all, that is, $M = 0$, this angle is exactly $\varphi_\infty = 0$. This is of course the case in which the light rays follow straight lines. However, in the presence of M, these rays are deflected by a total amount φ_∞. Note that the maximum possible deflection is set by the radius of the object. Indeed, $u(\varphi)$ has a maximum at $\varphi_{\rm max} = \pi/2 - 2M/b + \mathcal{O}(M^2)$, from which we can extract the value of r at which the rays get closer to the star. It turns out to be $r = b - M + \mathcal{O}(M^2)$. This implies, in particular, that any ray approaching an object with radius r_0 and whose impact parameter is $b \lesssim r_0 + M$ will not be able to escape to infinity, but will fall into it. Thus, the maximum deflection is

$$\varphi_{\infty,\rm max} = -\frac{4M}{r_0 + M}\,. \tag{3.58}$$

For the Sun, $r_0 = 6.96 \times 10^5\,$km, while $M = 1.47\,$km, in which case $\varphi_{\infty,\rm max} = 1.74''$. This effect was claimed to be observed during the lunar eclipse of the 29th of May, in 1919 by the team of Sir Arthur Stanley Eddington and considered the first tested prediction of Einstein's theory of gravity. Observations were made from the West African island of Príncipe and from from Sobral, Brazil. There has been some suspicion over how trustful of the results of this first measurement, and turns out to be a very interesting event in the history of physics, see for example [6].

Finally, let us mention that bending of light rays near more massive objects is sometimes referred to as *gravitational lensing*. In this case, the metric is deformed so much that the same object can be seen several times, as its light rays are bend from both sides of the massive object. See this NASA webpage to know more and see a picture of this phenomenon taken with the Hubble telescope.

3.6 Black holes

3.6.1 Coordinate and physical singularities

We have so far studied the first corrections that the new theory of gravity explains and predicts related to corrections to Newtonian gravity. Let us now examine the Schwarschild metric and try to make sense of it accross $r = 2M$. It is clear, as we already pointed out, that the line element Eq. (3.6),

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 - \frac{2M}{r}\right)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) .$$

has a problem for this particular value of the radial coordinate, since some of the components of the metric vanish or blow up. This problem could either be physical or not. Actually, we shall argue that it is not, but just related to our choice of coordinates. This situation is analogous, for example, to the case in which we parameterise Euclidean, flat,three-dimensional space in spherical coordinates. Indeed, the line element becomes

$$\mathrm{d}s^2_{\mathbb{R}^3} = \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) , \tag{3.59}$$

which has problems at the origin, $r = 0$; and at the poles, $\theta = 0\,,\pi$. Of course, this is a consequence of the choice of coordinates, since in Cartesian coordinates the metric is just the identity, regular at all points. Then, whenever a singularity can be removed by an approppriate change of coordinates (*i.e.*, a diffeomorphism), we will refer to it as *coordinate singularity*, as they have no physical meaning.

In contrast, *physical singularities* may occur. In this case, we would find some diffeomorphism invariant quantity such as $R\,, R^{\rho\sigma}R_{\rho\sigma}$ or even $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ that behaves badly at the singularity.

Let us then discuss whether the singularity at $r = 2M$ is physical or not. First of all, we shall ask if any observer can reach it in finite time. Unfortunately for them, radially infalling ($\theta = \pi/2$, ϕ =constant) observers will actually do so. Indeed, consider the variation of the radial position with respect to proper time given by Eq. (3.29). Say they are initially ($\tau = 0$) at $r = R$ and starts following its geodesic. From the equation we get that

$$\mathrm{d}\tau = -\frac{\mathrm{d}r}{\left(\hat{E}^2 - 1 + \frac{2M}{r}\right)^{\frac{1}{2}}} \tag{3.60}$$

For simplicity, we consider the case of an observer whose energy corresponds to being at rest at infinity ($\hat{E} = 1$). Then, Eq. (3.60) can be integrated easily and we learn that in the reference frame of the infalling observer it takes

$$\Delta\tau = -\int_R^{2M}\frac{\mathrm{d}r}{\sqrt{2M/r}} = \frac{4M}{3}\left(\frac{r}{2M}\right)^{\frac{3}{2}}\Bigg|_{2M}^{R} < \infty \tag{3.61}$$

to reach the surface $r = 2M$.

How much coordinate time does it take for them to cross it? Using the fact that energy is conserved, we can write $\mathrm{d}t/\mathrm{d}\tau = g^{tt}P_t/m = (1-2M/r)^{-1}\hat{E}$; so

$$\mathrm{d}t = \frac{\hat{E}\mathrm{d}\tau}{1-\frac{2M}{r}} = -\frac{\hat{E}\mathrm{d}r}{\left(1-\frac{2M}{r}\right)\sqrt{\hat{E}-1+\frac{2M}{r}}}\,. \tag{3.62}$$

We consider again the case $\hat{E}=1$. Writing this last expression in terms of the distance to $r=2M$, which we call $\epsilon = r-2M$, we get

$$\mathrm{d}t = -\frac{(\epsilon+2M)^{\frac{3}{2}}}{\epsilon\sqrt{2M}}\mathrm{d}\epsilon \simeq \left(-\frac{2M}{\epsilon}+\mathcal{O}(\epsilon^0)\right)\mathrm{d}\epsilon\,. \tag{3.63}$$

In particular, t grows as $\log\epsilon$ as $\epsilon\to 0$. This is interesting, since t can be understood as the time measured by a distant observer. Actually, consider an observer at rest at $(t,r,\theta,\varphi) = (t(\tilde{t}), r_0, \pi/2, \varphi_0)$, with r_0 and φ_0 constants and $\tilde{t}$ is its proper time[7]. The velocity of this observer is $\mathrm{d}/\mathrm{d}\tilde{t} = U^\alpha\partial/\partial x^\alpha = U^t\partial/\partial t$ and since $U^\mu U_\mu = -1$,

$$-1 = g_{\mu\nu}U^\mu U^\nu = g_{tt}(U^t)^2\,, \;\Rightarrow\; U^t = \frac{1}{\sqrt{1-\frac{2M}{r_0}}}\,. \tag{3.64}$$

In particular, $\mathrm{d}/\mathrm{d}t \to \mathrm{d}/\mathrm{d}\tilde{t}$ as $r_0\to\infty$. Then t can be understood as the proper time for a distant observer. For instance, imagine our infalling observer carries a laser with them and sends a pulse every $\Delta\tau$. According to Eq. (3.62), the time delay between the emission of two pulses measured at infinity is

$$\Delta t = \frac{\hat{E}}{1-\frac{2M}{r}}\Delta\tau\,. \tag{3.65}$$

So, from the point of view of the observer at infinity, the pulses are emitted every Δt, which diverges as $r\to 2M$.[8]

These results mean that, even though our infalling observer crosses $r=2M$ in a finite proper time, from infinity it seems it takes them for ever to do so. Anyway, for the purposes of our current discussion, this just reinforced our believe that the coordinates in terms of which or metric is written are not appropriate to describe the region near $r=2M$.

Is there a physical singularity at that point? We know that $R=0$ and $R_{\rho\sigma}=0$. We could also compute $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 48M^2/r^6$. So nothing problematic seems to be found at $r=2M$. But if the sungularity is not physical, we should be able to find coordinates that behave smoothly on $r=2M$. This is precisely what we are going to do next.

[7]Note that this observer is not following a geodesic, so there must exist some force keeping it at this point; for example, it is in a spacecraft with its engine turned on.

[8]Note that Δt is however not the time elapsed between the detection of the two pulses, as the second pulse will have to travel more distance since it will be emitted from a position closer to $r=2M$.

3.6.2 The Kruskal–Szekeres coordinates

For $r > 2M$, let's consider the change of coordinates

$$\begin{aligned} X &= \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right) , \\ T &= \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) . \end{aligned} \tag{3.66}$$

Interestingly, the metric Eq. (3.6) in terms of them becomes

$$\mathrm{d}s^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} \left(-\mathrm{d}T^2 + \mathrm{d}X^2\right) + r^2 \mathrm{d}\Omega_2 , \tag{3.67}$$

where $r = r(X,T)$ is the expression of the previous radial coordinate in terms of the new ones, obtained by inverting Eq. (3.66). Note that the new line element is perfectly regular at $r = 2M$. From the change of coordinates in Eq. (3.66), we can also write

$$X^2 - T^2 = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} , \qquad \frac{T}{X} = \tanh\left(\frac{t}{4M}\right) \tag{3.68}$$

Thus, constant t slices correspond to straight lines with slope $\tanh(t/4M)$ in the (X,T)-plane. Conversely, constant r slices correspond to hyperbolas. Furthermore, a photon moving in the radial direction (*i.e.* constant θ and φ) has

$$\frac{\mathrm{d}T}{\mathrm{d}X} = \pm 1 , \tag{3.69}$$

which means that light cones are represented by 45° lines.

Since in Eq. (3.67) we have got rid off the coordinate singularity at $r = 2M$, even though we started in the region $r > 2M$, using Eq. (3.68) we can extend the new coordinates X and T to the region where $r < 2M$. There we will have

$$\begin{aligned} X &= \left(1 - \frac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) , \\ T &= \left(1 - \frac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right) . \end{aligned} \tag{3.70}$$

Now we can try to make sense of the full (X,T)-plane, see Fig. 8. The diagonals, corresponding to $r = 2M$ and $t = \pm\infty$, divide the (X,T)-plane in four distinct sectors, whose physics we shall discuss next.

- **Sector I.** This is the region where $r > 2M$ which was covered completely by our initial parametrization and whose metric was found in Eq. (3.6). Here, slides of constant r are spacelike and represented by hyperbolas. Something interesting occurs when these hyperbolas approach the diagonal. Indeed, imagine that an observer who is at rest at some $r_1 > 2M$ sends a message to their colleague, who is at rest at $r_2 > r_1 > 2M$. We discussed already how their velocities look like in this situation (see Eq. (3.64)),

$$U^t_{(1)} = \frac{1}{\sqrt{1 - \frac{2M}{r_1}}} , \qquad U^t_{(2)} = \frac{1}{\sqrt{1 - \frac{2M}{r_2}}} . \tag{3.71}$$

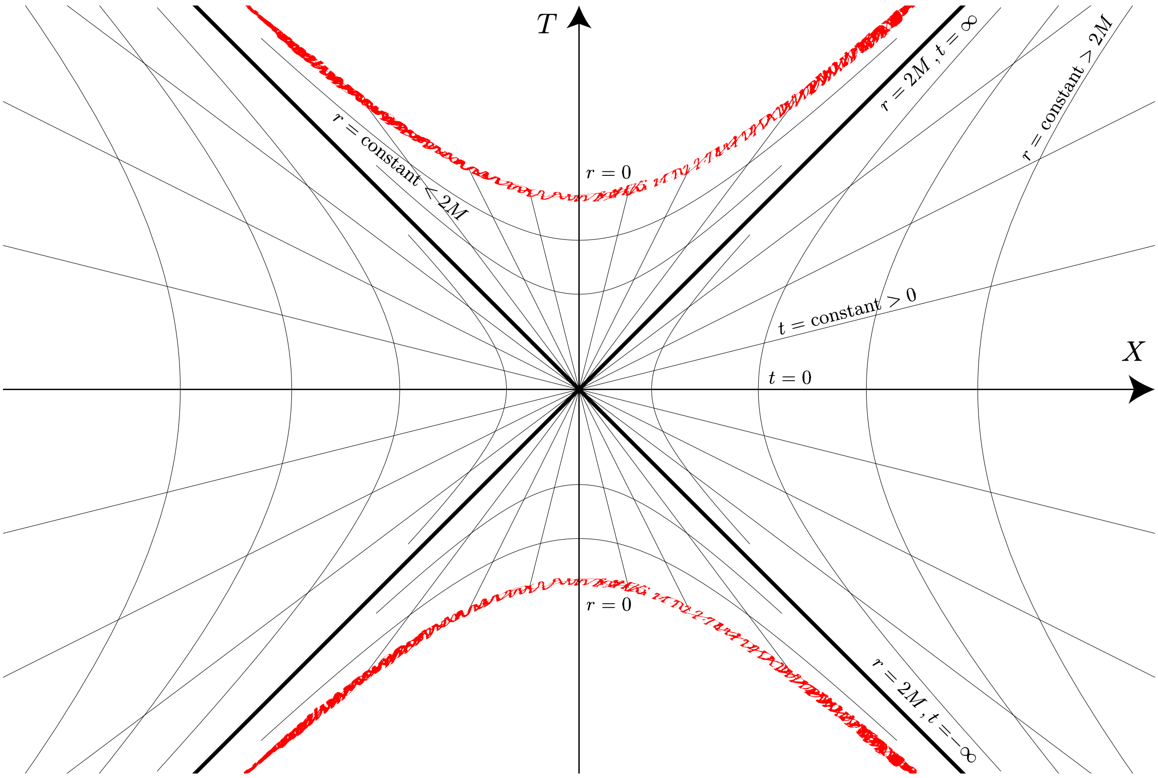

Figure 8: Schwarschild spacetime in Kruskal–Szekeres coordinates. The physical singularities are represented by the red brushed curves, corresponding to $r = 0$ in the original coordinates. Note these singularities are spacelike.

To send the message, light with a certain frequency ω_1 is used. The photon send will have a momentum K^α, and the frequency measured by the first observer (which corresponds to the frequency of the device, as they are holding it in their own system of reference) is given by $K_\alpha U^\alpha_{(1)} = \hbar\omega_1$, as this is how they would measure it in flat space. When the photon is received, on the other hand, they will measure $K_\alpha U^\alpha_{(2)} = \hbar\omega_2$. Taking advantage of $\partial/\partial t$ being a Killing vector, we can find the relation between the two frequencies to be

$$\frac{\omega_2}{\omega_1} = \left(\frac{1 - \frac{2M}{r_2}}{1 - \frac{2M}{r_1}}\right)^{\frac{1}{2}} . \tag{3.72}$$

We can say then that light shifts to smaller frequencies as it climbs the gravitation field. This phenomenon is know as *gravitational redshift*. For a very distant observer ($r_2 \to \infty$), it becomes

$$\omega_\infty = \left(1 - \frac{2M}{r_1}\right)^{\frac{1}{2}} . \tag{3.73}$$

One final observation is that ω_∞ vanishes as the first observer is closer to $r = 2M$. In a sense, the geometry is describing something *black*: objects look fainter the closer they are to $r = 2M$. Additionally, we know that they will cross this slice if they are allowed to fall freely. What will happen next?

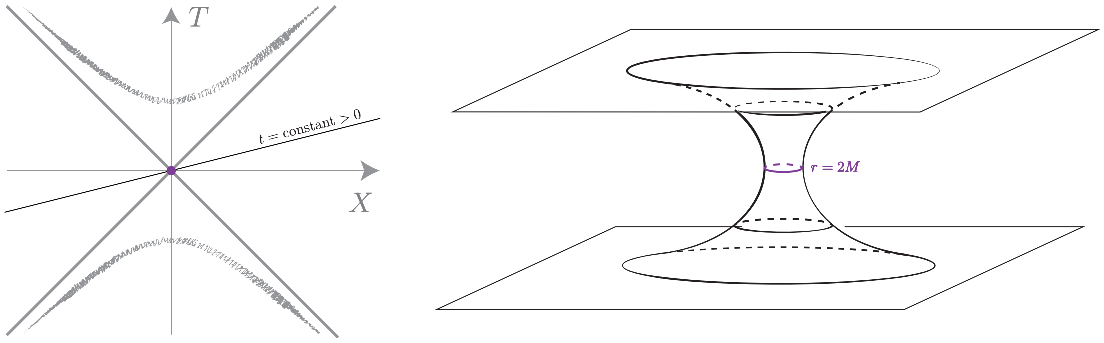

Figure 9: The spaces $t =$ constant correspond to wormhole geometries. They connect two causally disconnected universes.

- **Sector II** . Let us now examine the mysterious region with $r < 2M$. First of all, we note that any signal sent from there cannot be accessed by an observer moving in **Sector I**, since for that such signal would need to go faster than light at some point. In that sense, the surface $r = 2M$ is a *horizon*. In particular, neither photons not particles can escape from this region once they have entered, and in this precise sense this geometry is a *hole*. Now you understand why we call this geometry a "black hole".

  Moreover, note that unlike in the previous case, where $r > 2M$, in this part of the diagram the hyperbolas corresponding to constant-r slices are spacelike, whereas constant-t lines are timelike. This means that any object will be bound to advance in the r coordinate, while they could in principle move along a constant t slice. Put yet in a more poetical way, inside the horizon, time becomes space and space become time.

  Remarkably, the singularity found at $r = 0$ is also spacelike. For this reason, it is not a point in space but a moment in time. In particular, black holes do not have a center. Actually, the singularity in this region of the diagram is by no means observable as geodesics finish at $r = 0$ (where the curvature singularity is found): there is nothing to future past of the singularity.

- **Sector III.** Here, spacetime has properties similar to those of **Sector I**. However, it is causally disconnected from it: to send information from one of these regions to the other, one should do it across the horizon and needs a velocity faster than the speed of light. This region then is actually describing a disconnected region of spacetime, connected to **Sector I** through a wormhole, which correspond to taking a whole $t =$ constant slice. This is represented in Fig. 9.

  In our geometry this wormhole cannot be crossed as nothing travels faster than the speed of light. There has been however raising interest in trying to build traversable wormholes by incormporating matter fields and wuantum effects in the game, see Refs. [7–9].

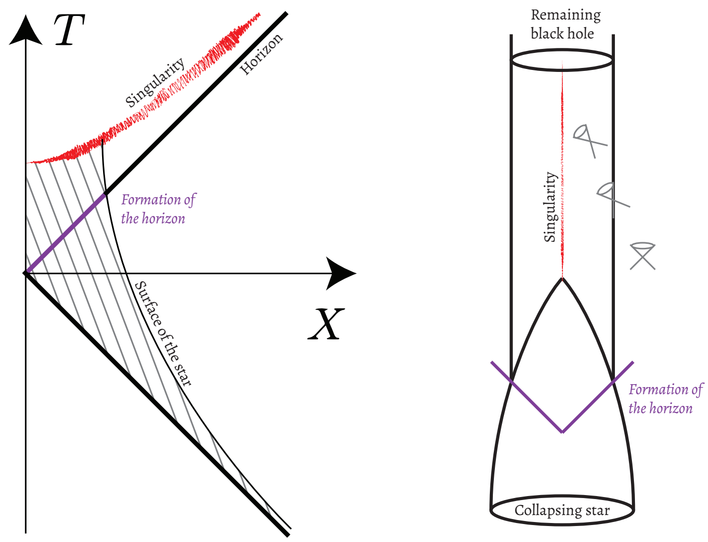

Figure 10: **Right:** Illustration of the spacetime diagram of a collapsing star. In this scenario, only some parts of the first two regions described in the main text are physical. **Left:** Pictorial representation of a collapsing star. Some light cones have been depicted in grey.

- **Sector IV.** This region is analogous to the second region we analysed. In this case, however, the singularity is in the past of all observers, so we could refer to it as a *white hole*. Information from this region could be detected in **Sector I** after it leaves this part of the diagram through the "past horizon" corresponding to $r = 2M$ and $t = -\infty$.

  One could be tempted to identify the *white hole* with the Big Bang. This is not quite the case, although it can serve as an analogy. Indeed, Big Bang singularity appears when we consider the cosmological evolution of the Universe; where matter fields play a crucial role. In contrast, here we are discussing vacuum solutions to Einstein's equations.

We could wonder how physical **Sectors II** and **IV** are, since we see that they lead to strange phenomenology (wormholes and white holes have not been observed so far). Actually, we know that these two regions are not realised when the black hole is formed from the collapse of very massive stars. Indeed, during the collapse of a star we only find parts of **Sector I** and **II**, as inside the star the metric is not described by the Schwarschild metric anymore. This is more easily understood by looking at Fig. 10. Before the horizon appears, the metric is perfectly fine in the original coordinates and Schwarschild metric only models spacetime outside the star.

If you are curious about it, please visit this webpage from Andrew Hamilton to examine how it is to fall into a black hole.

3.7 Beyond spherical symmetry: the Kerr solution

3.7.1 The Kerr black hole

We finish this chapter by presenting the Kerr solution. This is a geometry that solves Einstein's vacuum equations and describes a rotationg black holes. It reads

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{\Sigma}\right)dt^2 + \frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\varphi^2 + \Sigma\left(\frac{dr^2}{\Delta}+d\theta^2\right) \\ &- 2a\sin^2\theta\frac{2Mr}{\Sigma}dtd\varphi\,, \end{aligned} \tag{3.74}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,, \qquad \Delta = r^2 - 2Mr + a^2 + Q^2\,. \tag{3.75}$$

Let us try to describe the most salient features of this spacetime. First of all, it is distinct from Schwarschild metric, as it is not spherically symmetric. Still, spherical symmetry is restored by setting $a = 0$, in which case Schwarschild spacetime is recovered. Actually, the parameter a is related to the fact that in the Kerr solution spacetime itself is rotating, as we shall argue later. The corresponding spin is

$$J = Ma\,. \tag{3.76}$$

Thus, this spacetime possesses *axial symmetry* about the rotation axes. As a manifestation, $\partial/\partial\varphi$ is a Killing vector. As in the Schwarschild-metric, $\partial/\partial t$ is also a Killing vector and, in particular, this spacetime is stationary in the region outside the horizon we will find later. However, it is not static as it is not possible to get rid of the $g_{t\varphi}$ component as long as $a > 0$.

On the other hand, this metric is also asymptotically flat. Actually, analysing the asymptotic region $r \to \infty$, we see that

$$g_{tt} = -1 + \frac{2M}{r} + \mathcal{O}(r^2)\,. \tag{3.77}$$

and comparing Eq. (3.77) to the Schwarschild solution we conclude that M is the mass of this spacetime[9]. Note that there is not an equivalent to Birkhoff theorem (**Theorem 1**) for rotating spacetimes. That is to say, the exterior solution of the spacetime corresponding a rotating star is not unique, as suggested by Fig. 11. However, there is a theorem by Hawking, Carter and Robinson that states

[9] One may feel uneasy before the fact that this seems a coordinate dependent statement. Actually, the mass M and spin J of a given spacetime can be properly defined using the so called *Hamiltonian formulation of gravity*. They would read (see [10], page 147)

$$M = -\frac{1}{8\pi}\lim_{A\to\infty}\oint_{S_A}(K-K_0)\sqrt{\gamma}d^2\theta\,, \qquad J = -\frac{1}{8\pi}\lim_{A\to\infty}\oint_{S_A}(K_{ij}-Kh_{ij})\varphi^i r^j\sqrt{\gamma}d^2\theta\,. \tag{3.78}$$

Here, the integral is performed over two-spheres S_A of area A, which eventually is taken to infinity. Additionally, K_{ij} corresponds to the extrinsic curvature of those spheres, K is its trace and K_0 is the corresponding value in Minkowski. Finally, φ^i is the Killing vector associated to rotational symmetry and r^j is the unit normal vector to the spheres.

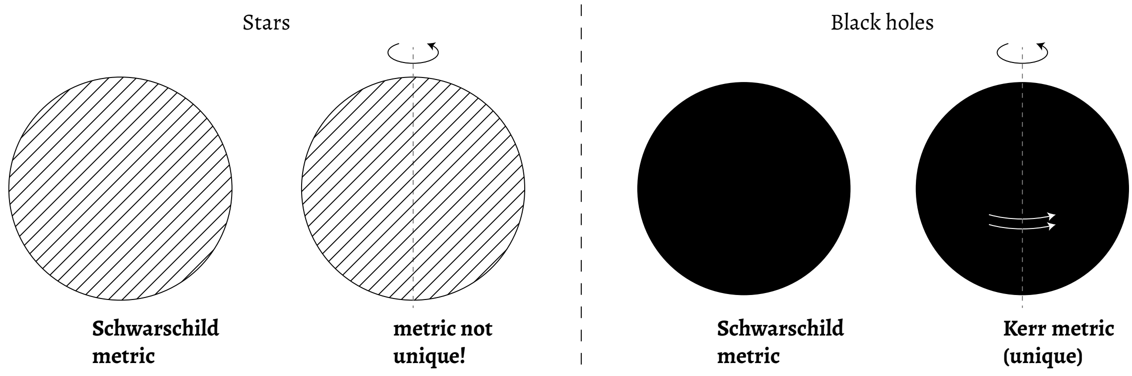

Figure 11: Visualization of the meaning of Birkhoff theorem and Hawking–Carter–Robinson theorem regarding the uniqueness of vacuum solution to Einstein's equations.

Theorem 2. *(Uniqueness of the Kerr black hole). Kerr spacetime Eq.* (3.74) *is the unique stationary, assymptotically free, vacuum solution to Einstein's equations which is regular on an outside a (non-degenerate) event horizon. There is a two-dimensional parameter family of such solutions, characterised by their mass M and angular momentum J.*

Put differently, any other stationary, asymptotically-free, vacuum solution which is not related to the Kerr solution via a diffeomorphism, will contain some pathology. Consequently, it would be valid as an exterior solution for a rotating star if the surface of the star is found before the pathology appears. But it will not be a valid solution of pure empty space. Thus, this theorem is telling us that any rotating black hole is descrived by the Kerr solution in Eq. (3.74) and characterised by the two parameters M and J.

3.7.2 Dragging of intertial frames and the *ergosphere*

What is it that is rotating in the Kerr geometry? Clearly, as this is a vacuum solution, we should argue that it is spacetime itself. Indeed, consider the two conserved quantities along the trajectory of a particle with mass m, corresponding to the Killing vector fields $\partial/\partial t$ and $\partial/\partial\varphi$. As in Eq. (3.26),

$$g_{\alpha\beta}\xi^{\alpha}_{(1)}P^{\beta} = P_t =: -\hat{E}m\,, \qquad g_{\alpha\beta}\xi^{\alpha}_{(2)}P^{\beta} = P_{\varphi} =: -\hat{L}m\,. \tag{3.79}$$

We will again refer to them as the energy and angular momentum for obvious reasons. Note the appearance of the off-diagonal term of the inverse of the metric when we raise indexes,

$$\begin{aligned} U^t &= m^{-1}P^t = -g^{tt}\hat{E} + g^{t\varphi}\hat{L}\,, \\ U^{\varphi} &= m^{-1}P^{\varphi} = g^{t\varphi}\hat{L} - g^{\varphi\varphi}\hat{E}\,. \end{aligned} \tag{3.80}$$

Interestingly, the variation of the angle φ respect to coordinate time is generically different from zero. Actually, even in the case in which the particle has zero angular momentum $\hat{L} = 0$, the derivative of φ becomes

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}\varphi/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau} = \frac{U^t}{U^{\varphi}} = \frac{g^{t\phi}}{g^{tt}} =: \omega(r,\theta) = \frac{2Mra}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}\,. \tag{3.81}$$

Therefore, such a particle will be rotating with respect to an observer at infinity, even though its angular momentum is zero. The interpretation is precisely that it is spacetime itself, in which the particle is travelling, which is rotating. The particles following geodesics are bound to rotate together with it. This effect is suggestively referred to *dragging of inertial frames*. It also originates a gyroscopic precession called the *Lense–Thirring effect*.

This phenomenon becomes more and more violent as we approach to the center of the geometry. To inspect this, imagine that we keep a particle at constant r and θ coordinates, while allowing it to rotate in the $\partial/\partial\varphi$ direction. Generically, this will not be a geodesic. If its angular velocity as seen from infinity

$$\Omega := \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{U^\varphi}{U^t}\,. \tag{3.82}$$

is uniform, we can thing of this observer as being *stationary*. It is the analogous to the *static* observers, or observers at rest, that we considered in the Schwarschild geometry. The stationary particle's quadri-velocity can be written as

$$U = U^t\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\right) \tag{3.83}$$

Clearly, the angular velocity in Eq. (3.82) cannot be arbitrary, as particles cannot travel faster than light. In particular, $U^\mu U_\mu = -1$ implies that

$$g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi} < 0\,. \tag{3.84}$$

Therefore, the angular velocities of stationary observers are constrained by

$$\Omega_{\min} \;<\; \Omega \;<\; \Omega_{\max}\,, \tag{3.85}$$

where

$$\Omega_{\min} = \omega - \sqrt{\omega^2 - g_{tt}/g_{\varphi\varphi}}\,, \qquad \Omega_{\max} = \omega + \sqrt{\omega^2 - g_{tt}/g_{\varphi\varphi}}\,; \tag{3.86}$$

and $\omega = -g_{t\varphi}/g_{tt}$ actually coincides with Eq. (3.81) found earlier while discussing the dragging of inertial frames. Note $\omega = (\Omega_{\min} + \Omega_{\max})/2$.

There are several interesting things to note here. First, consider the surface where g_{tt} vanishes, given by

$$g_{tt} = 0\,, \quad r = r_e(\theta) := M + \sqrt{M^2 - a^2\cos^2\theta}\,. \tag{3.87}$$

We refer to this surface as the *ergosphere*. On the ergosphere, $\Omega_{\min}$ vanishes and for smaller values of r it becomes positive. This means that inside this surface all stationary observers must rotate with positive angular velocity. In particular, static observers only exist outside the ergosphere. Additionally, the allowed range of angular velocities Ω depends on r in such a way that and $\Omega_{\min}$ coalesces with $\Omega_{\max}$ when $\Delta = 0$ ($g_{rr} \to \infty$) at

$$r = r_+ = M + \sqrt{M^2 - a^2}\,. \tag{3.88}$$

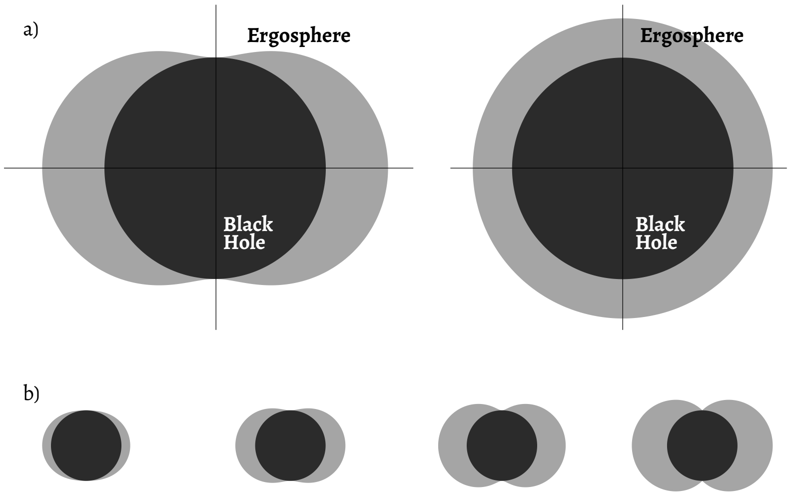

Figure 12: a) Sections of the ergosphere and event horizon of a Kerr black hole; as seen from the equatorial plane (left) and from the top (right). b) Different shapes of the ergosphere as we increase the parameter $|a|$. The rightmost picture corresponds to an extremal Kerr solution, with $M = |a|$. These pictures are normalised so that they have the same r_+. Thus, the mass of these black holes increase to the right.

For smaller values of r, there cannot be stationary observers. Gravitational pull has become too strong and there is no escape from it anymore: the value of r in Eq. (3.88) corresponds to an event horizon. The critical value of Ω when $\Omega_{\min} = \Omega_{\max}$ at the horizon

$$\Omega_{\mathrm{H}} = \frac{a}{r_+^2 + a^2} \tag{3.89}$$

can be interpreted as the angular momentum of the horizon itself. In Fig. 12 we show the different parts of the Kerr black hole and examine how they depend on a.

3.7.3 Singularities in the Kerr black hole

Kerr metric in Eq. (3.74) seems to be problematic in two different loci. When $\Sigma = 0$, the determinant of the metric $\det g_{\mu\nu} = -\Sigma^2 \sin^2\theta$ vanishes; while when $\Delta = 0$, we have seen that there is a horizon. While the latter is just a coordinate singularity that can be cured by an appropriate change of variables, at $\Sigma = 0$ the Kretschmann scalar diverges. This is then a physical curvature singularity, found when $r = 0$ and $\theta = \pi/2$. Here, we could be fooled by our spherical symmetry intuition and think that $r = 0$ corresponds to a point in spacetime. If this was the case, the extra requirement $\theta = \pi/2$ would seem odd. Actually, in this case the singularity is not a point but a ring (thus, it is not zero-dimensional but one-dimensional). This can be easily understood by studying the limiting case were $M = 0$. In this case the metric becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \Sigma\left(\frac{\mathrm{d}r^2}{r^2 + a^2} + \mathrm{d}\theta^2\right) + (r^2 + a^2)\sin^2\theta \mathrm{d}\varphi^2\,. \tag{3.90}$$

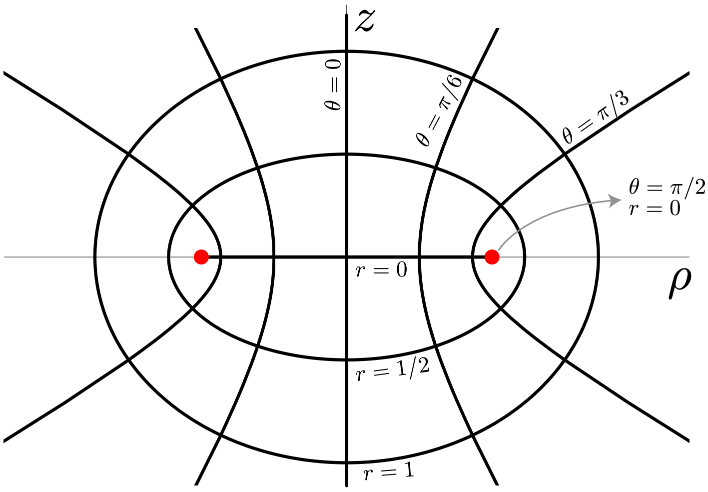

Figure 13: Ring singularity inside a Kerr black hole represented in terms of the coordinates z and ρ. The singularity is found at the ring were $\theta = \pi/2$ and $r = 0$, represented by the red dots. The disc $r = 0$ separates two physically distinct regions were $r < 0$ and $r > 0$ as explained in the main text.

The change of coordinates $z = r\cos\theta$, $\rho = \sqrt{r^2 + a^2}\sin\theta$, reveals that Eq. (3.90) is nothing but an involved way of writing Minkoswi space, as it becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2\,. \tag{3.91}$$

However, now it is transparent that $r = 0$ is a disk, while $r = 0$ and $\theta = \pi/2$ corresponds to its boundary, which is a ring (see Fig. 13).

Even though we have discussed this with $M = 0$ for simplicity, the singularity has also ring topology for finite mass. Moreover, something interesting happens for non-vanishing mass. Note that the metric ceases to be symmetric under the change $r \to -r$. For this reason, passing through the ring singularity (i.e. crossing the disk $r = 0$) would bring us to a distinct region where $r < 0$. At the same time, near the singularity $g_{\varphi\varphi} \simeq 2Ma^2/r$, and so it changes sign across the disk. Consequently, $\partial/\partial\varphi$ becomes timelike in the region where $r < 0$, and so it leads to the presence of timelike closed curves there. Thus, the Kerr metric contains time machines: we should understand whether this is problematic.

Consider the case when $a^2 < M^2$. Note that there are actually two horizons, since $\Delta = 0$ occurs at

$$r = r_\pm = M \pm \sqrt{M^2 - a^2}\,. \tag{3.92}$$

The outer horizon r_+ was introduced already in Eq. (3.88), while now we see there is also an *inner horizon* at $r = r_- < r_+$. In this case, both the singularity and the time machines are covered by the horizon and thus we do not have to worry about them since they are not observable from infinity.

On the other hand, if $a^2 < M^2$, the values of $r_\pm$ in Eq. (3.92) become complex. This means that there are not horizons at all and thus the ring where $\Sigma = 0$ becomes a *naked singularity*, in the sense that it can be observed from infinity.

Naked singularities are bad. Imagine we are simulating the gravitational collapse of a star and in the process we produce a region with diverging curvatures appear. The simulation would have to stop there, for new physics would be needed to resolve the singularity and the computation would go beyond the applicability of general relativity. Put differently, if a naked singularity occurs outside an event horizon, we need more than our current understanding of gravity to describe it. We would probably have to consider higher order curvature terms and/or quantum gravity effects. In contrast, if singularities are covered by an event horizon, the problem is avoid, as the region of spacetime observable from infinity would be perfectly described by Einstein's equations. Thus, if we want our spacetime to be *strongly asymptotically predictable*, all singularities should rather be enclosed by horizons.

On the other hand, it is commonly believed that naked singularities cannot be produced in Nature. This expectation is stated in the form of the *cosmic censorship conjectures*. Because it seems that Nature hides any break down of general relativity inside event horizons, we believe that any evolution of Einstein's equations will lead to black holes rather than naked singularities[10]. For us, this means that we discard the Kerr solution with $|a| < M$ as unphysical as it would contain a region not properly described within general relativity and also closed time curves reachable from infinity. We refer to the case $|a| = M$ as the extremal Kerr solution, which cannot be over-spun.

[10] Note, however, we can have an initial singularity, such as the Big Bang.

4 Gravitational waves

Since the first direct detection of gravitational waves coming from the mergers of two black holes on 14 September 2015 [11], the area of gravitational wave astronomy has become very active. In these few years, there have already been several remarkable observations. For example, gravitational waves coming from the merger of two neutron stars [12], combined with the γ-ray bursts detected almost at the same time, allowed the location on the sky of the event that produced them. The study of the electromagnetic counterpart of the merger was tracked during several days by many different telescopes around the globe [13]. By now, there have been almost 100 events detected by the LIGO-Virgo-KAGRA Collaboration, with detectors placed in the United States, Italy, and Japan. Thus, even though gravitational waves were already discussed by Einstein himself [14] soon after he proposed the equations for General Relativity, it has been only a century later that they have become experimentally accessible. In this section, we plan to describe them mathematically and understand their physics.

4.1 Linear approximation and symmetry considerations

We think of gravitational waves as small perturbations about Minkowski metric. This is to say that our space-time metric will be given, in certain coordinates x^α by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2)\,, \tag{4.1}$$

where $\eta = \text{diag}(-1,1,1,1)$. In Eq. (4.1) we are just stating that we neglect any term which is not linear in h, meaning that contains two or more components of $h_{\mu\nu}$. The goal is to examine how Einstein's equations look like in terms of $h_{\mu\nu}$, and, eventually, solve them. But we should first make a couple of remarks, related to coordinate transformations of the metric in Eq. (4.1).

- **Lorentz transformations**. Imagine we apply a Lorentz transformation to the original coordinate system x^α,

  $$y^\beta = \Lambda^\beta{}_\alpha x^\alpha\,. \tag{4.2}$$

  In particular, $\partial y^\beta/\partial x^\alpha = \Lambda^\beta{}_\alpha$. Therefore, in the new coordinate system y^β, the metric becomes

  $$\tilde{g}_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu\, \eta_{\rho\sigma} + \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu\, h_{\rho\sigma} + \mathcal{O}(h^2) = \eta_{\mu\nu} + \tilde{h}_{\mu\nu} + \mathcal{O}(h^2)\,, \tag{4.3}$$

  where we have used that $\tilde{\eta}_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu\, \eta_{\rho\sigma} = \eta_{\mu\nu}$, since the Minkowski metric is invariant under Lorentz transformations. Consequently,

  $$\tilde{h}_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu\, h_{\rho\sigma} \tag{4.4}$$

  transforms as a tensor under Minkowski transformations. This observation is useful, because it allows us to interpret the "radiation field" $h_{\mu\nu}$ as a tensor defined on flat space. Put differently, we can imagine that we are dealing with a tensor field defined on flat Minkowski space and forget about the curvature of spacetime itself.

■ **Diffeomorphism invariance**. Apart from the previous observation regarding Lorentz transformations, it is convenient to examine the impact that a coordinate transformation would have in Eq. (4.1). Actually, the results we are going to find in this discussion will turn out to be quite useful later.

If we want the metric to look like a small perturbation about the Minkowski metric, we need the coordinate transformation to be "small". We can thing of such a transformation as originated by a vector field ξ^α,

$$\tilde{x}^\alpha = x^\alpha + \xi^\alpha(x) + \cdots \,. \tag{4.5}$$

More precisely, given a vector field $X^\alpha(x)$, we could find its integral curves by solving the the differential equation $\mathrm{d}x^\alpha/\mathrm{d}\lambda = X^\alpha(x)$. Then, Eq. (4.5) is just telling that we are moving the coordinates along the integral curves as

$$\tilde{x}^\alpha = x^\alpha + \lambda\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} + \mathcal{O}(\lambda^2) = x^\alpha + \lambda X^\alpha(x) + \mathcal{O}(\lambda^2)\,, \tag{4.6}$$

with $\xi^\alpha = \lambda X^\alpha$. Thus the statement that ξ^α is small.

Next, we wish to understand what the metric looks like after performing such a change of coordinates. Note that

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta + \partial_\beta \xi^\alpha\,, \qquad \frac{\partial x^\alpha}{\partial \tilde{x}^\beta} = \delta^\alpha{}_\beta - \partial_\beta \xi^\alpha \,. \tag{4.7}$$

Therefore, the metric becomes

$$\tilde{g}_{\mu\nu} = (\delta^\rho{}_\mu - \partial_\mu\xi^\rho)(\delta^\sigma{}_\nu - \partial_\nu\xi^\sigma)(\eta_{\rho\sigma} + h_{\rho\sigma}) = \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu\,, \tag{4.8}$$

where, since we are interested in the linear perturbations, we are only keeping terms linear in ξ and h. In particular, we have

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu = h_{\mu\nu} - (\pounds_\xi \eta)_{\mu\nu}\,. \tag{4.9}$$

In conclusion, under an infinitesimal coordinate transformation generated by the vector field ξ, the metric perturbations transform as in Eq. (4.9). However, this coordinate transformation should not have any physical consequence, if we want our theory to be invariant under diffeomorphisms. Thus, we can refer to Eq. (4.9) as a *gauge transformation*, and we can use it to fix some components of the field $h_{\mu\nu}$ conveniently. We will take enormous advantage of such gauge transformations.

Let us finish this section making one remark. Sometimes, it could be that we are interested in computing linear perturbations about some reference metric $g^{(0)}_{\mu\nu}$ different from the Minkowski metric[11]. In this case the transformation on $h_{\mu\nu}$ would read similar to Eq. (4.9):

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - (\pounds_\xi g^{(0)})_{\mu\nu}\,. \tag{4.10}$$

[11]For example, to compute the quasinormal modes of black hole solutions.

4.2 Einstein's equations for linear perturbations

Now that we understand the different symmetry properties of the linear perturbations $h_{\mu\nu}$, let us write down Einstein's equations in terms of them. Keeping only the linear terms in $h_{\mu\nu}$, the Christoffel symbols become

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\right) . \tag{4.11}$$

From them, we find that the Riemann tensor reads

$$R_{\alpha\beta\mu\nu} = \eta_{\alpha\lambda}\partial_\mu \Gamma^{\lambda}_{\beta\nu} - \eta_{\alpha\lambda}\partial_\nu \Gamma^{\lambda}_{\beta\mu} = \frac{1}{2}\left(\partial_\mu\partial_\beta h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\mu\beta} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\nu\partial_\beta h_{\alpha\mu}\right) . \tag{4.12}$$

Interestingly, this expression is invariant under the gauge transformations from Eq. (4.9), $\tilde{R}_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}$. After computing the Ricci tensor and the Ricci scalar, we find that Einstein's tensor becomes

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \\ &-\frac{1}{2}\left(\partial^\alpha\partial_\alpha h_{\mu\nu} + \partial_\mu\partial_\nu h^{\alpha}{}_{\alpha} - \partial_\mu\partial_\alpha h^{\alpha}{}_{\nu} - \partial_\nu\partial_\alpha h^{\alpha}{}_{\mu} - \eta_{\mu\nu}\partial^\alpha\partial_\alpha h^{\beta}{}_{\beta} - \eta_{\mu\nu}\partial^\alpha\partial_\beta h^{\beta}{}_{\alpha}\right) . \end{aligned} \tag{4.13}$$

At this point, it is useful to define a tensor called "trace reverse" of $h_{\mu\nu}$, defined as

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\alpha}{}_{\alpha} . \tag{4.14}$$

Note that the trace of this new tensor is $\overline{h}^{\alpha}{}_{\alpha} = -h^{\alpha}{}_{\alpha}$, which justifies its name. In terms of it, Eq. (4.13) becomes

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{1}{2}\left(\partial_\alpha\partial^\alpha\overline{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\overline{h}^{\alpha\beta} - \partial_\mu\partial_\alpha\overline{h}^{\alpha}{}_{\nu} - \partial_\nu\partial_\alpha\overline{h}^{\alpha}{}_{\mu}\right) . \tag{4.15}$$

Note that this would simplify enormously if $\partial_\alpha\overline{h}^{\alpha\beta} = 0$. We wish to argue next that using the invariance under infinitessimal in Eq. 4.9 it is possible to work in a gauge where this is the case. Indeed, note that under a gauge transformation

$$\tilde{\overline{h}}_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\tilde{h}^{\alpha}{}_{\alpha} = \overline{h}_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + \eta_{\mu\nu}\partial_\alpha\xi^\alpha . \tag{4.16}$$

This implies that

$$\partial^\mu\tilde{\overline{h}}_{\mu\nu} = \partial^\mu\overline{h}_{\mu\nu} - \partial^\mu\partial_\mu\xi_\nu . \tag{4.17}$$

In particular, we can always change to a gauge where $\partial^\mu\tilde{\overline{h}}_{\mu\nu} = 0$, by choosing ξ such us it is a solution to

$$\partial^\mu\overline{h}_{\mu\nu} = \partial^\mu\partial_\mu\xi_\nu . \tag{4.18}$$

We refer to this choice of gauge as the *Lorentz gauge*, by its analogy to the electromagnetic case. On the other hand, note that it does not completely specify the gauge, as we could still add to ξ^α any vector χ^α such that $\partial_\mu\partial^\mu\chi^\alpha = 0$, for if we do so Eq. (4.18) would still be satisfied. We will use this remaining freedom in the next section. So far, going back to Eq. (4.15) we can conclude that the linearised Einstein's equations read

$$\partial_\alpha\partial^\alpha\overline{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{4.19}$$

in the Lorentz gauge, for which $\partial^\mu\overline{h}_{\mu\nu} = 0$ and $\partial^\mu\partial_\mu\xi_\nu = 0$.

4.3 Gravitational waves: plane wave solution

In the same way that electromagnetic waves are solutions to Maxwell's equations in the absence of charge or currents, gravitational waves are solutions to Einstein's equations in the absence of matter. Therefore, we now want to solve the linearised equations in Eq. (4.19) with $T_{\mu\nu} = 0$,

$$\partial_\alpha \partial^\alpha \overline{h}_{\mu\nu} = 0\,, \tag{4.20}$$

where we have to remember that these are written in the Lorentz gauge, $\partial^\mu \overline{h}_{\mu\nu} = 0$ and $\partial^\mu \partial_\mu \xi_\nu = 0$. Note that Eqs. (4.20) are just wave equations for the components of $\overline{h}_{\mu\nu}$, which means that they are solved by

$$\overline{h}_{\mu\nu} = \mathrm{Re}[A_{\mu\nu} \exp(i k_\beta x^\beta)]\,, \tag{4.21}$$

with k_β some one-form and $A_{\mu\nu}$ some symmetric $(0, 2)$ tensor. Moreover, Eq.(4.20) imposes that $k_\beta k^\beta = 0$, which implies that k^β is a light-like vector and, consequently, gravitational waves travel at the speed of light.

At first sight, it would seem that the symmetric tensor $A_{\mu\nu}$ has $n(n+1)/2 = 10$ degrees of freedom, but this is not true because gauge fixing conditions fix many of them. For instance, $\partial^\beta \overline{h}_{\alpha\beta} = 0$ implies that

$$k^\beta A_{\alpha\beta} = 0\,. \tag{4.22}$$

In words, the tensor $A_{\alpha\beta}$ is orthogonal to k^β.

Performing a rotation of the axes, we can assume without loss of generality that the gravitational wave propagates along the $x^3 = z$ direction. Then $k^\alpha = (k, 0, 0, k)$ and Eq. (4.22) implies $A_{t\mu} = A_{z\mu}$. The last gauge fixing condition, $\partial^\mu \partial_\mu \xi_\nu = 0$, is satisfied by the forms $\xi_\nu = B_\nu \exp(k_\beta x^\beta)$. These can be used to further demand that $A_{tx} = A_{ty} = 0$ and that the trace vanishes, $A^\alpha{}_\alpha = 0$. Now there is no gauge freedom left, and defining $A_{xx} = A_+$ and $A_{xy} = A_\times$ the tensor $A_{\mu\nu}$ becomes

$$A^{\mathrm{TT}}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \tag{4.23}$$

We refer to this complete gauge fixing as the *transverse-traceless gauge* (TT). Notice that tracelessness implies that $h^{\mathrm{TT}}_{\mu\nu} = \overline{h}^{\mathrm{TT}}_{\mu\nu}$ in this gauge.

In the end, we see that there are only two remaining propagating degrees of freedom, corresponding to A_+ and $A_\times$. The full solution can be written in terms of two polarization vectors $\mathbf{e}_+ = \mathrm{d}x \otimes \mathrm{d}x - \mathrm{d}y \otimes \mathrm{d}y$ and $\mathbf{e}_\times = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$ and reads

$$h^{\mathrm{TT}} = \mathrm{Re}\left[(A_+ \mathbf{e}_+ + A_\times \mathbf{e}_\times) \exp\left(-ik(t-z)\right)\right]\,. \tag{4.24}$$

Now that we have characterised gravitational waves, we should investigate how they act on text particles. Eventually, that is what we need if we want to build a gravitational wave detector.

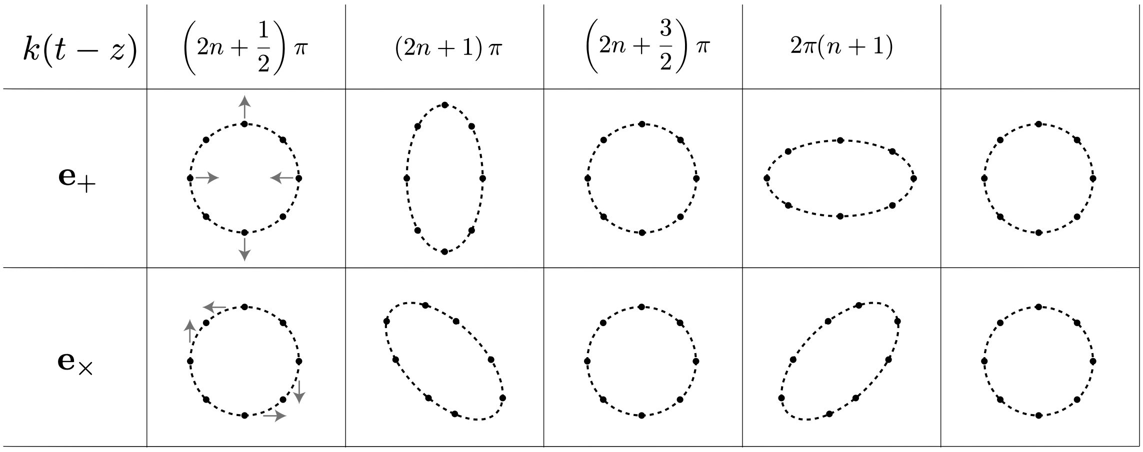

Figure 14: Effect of the two polarization modes of gravitational waves in test particles distributed in the (x, y)-plane. The direction to which the particles will move is represented in the first column (arrows stand for the velocity of the particles, not to their acceleration computed in Eq. (4.27)).

4.4 Effect of gravitational waves on test particles

Consider a test particle at rest in the spacetime Eq. (4.1), where $h_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu}$ is the solution to vacuum Einstein's equations, given by Eq. (4.24) in the appropriate gauge discussed in the previous section. Note that $\mathrm{d}/\mathrm{d}\tau = (1, 0, 0, 0) = \partial/\partial t$ is a solution to the geodesic equation

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{x^\nu}{\mathrm{d}\tau} = 0\,. \tag{4.25}$$

since $\Gamma^\alpha_{tt} = 0$. Thus, a single particle does not sense the effect of the gravitational wave passing by. This is somehow a consequence of the equivalence principle: what matters in general relativity is not the absolute state of movement but the relative acceleration between objects.

Thus, let us take two such test particles following the geodesics generated by $U = \partial/\partial t$. Let $\xi = \xi^i\partial/\partial x^i$ be the *deviation vector* between two particles. In particular $\pounds_U\xi = 0$, and we can use the deviation equation derived in Section 2.4, from which, using Eq. (4.12),

$$\frac{\mathrm{D}^2\xi^k}{\mathrm{d}\tau^2} = R^k{}_{ttm}\xi^m = -\delta^{kj}R_{jtmt}\xi^m = \frac{1}{2}\partial_t\partial_t h^{\mathrm{TT}}_{jk}\,. \tag{4.26}$$

Finally, since at leading order in $h_{\mu\nu}$ the proper time is $\mathrm{D}/\mathrm{d}\tau = \partial/\partial t$, we get

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix}\xi^x\\ \xi^y\end{pmatrix} = \frac{k^2}{2}\begin{pmatrix}\xi^x\\ -\xi^y\end{pmatrix}\mathrm{Re}[A_+e^{-ik(t-z)}]\,,\quad \frac{\partial^2}{\partial t^2}\begin{pmatrix}\xi^x\\ \xi^y\end{pmatrix} = \frac{k^2}{2}\begin{pmatrix}\xi^y\\ \xi^x\end{pmatrix}\mathrm{Re}[A_\times e^{-ik(t-z)}]\,, \tag{4.27}$$

for the $\mathbf{e}_+$ and $\mathbf{e}_\times$ modes, respectively. Note that ξ^z remains unaffected. The result in Eq. (4.27) is represented in Figure 14, where the vector ξ can be thought of as the separation from the center of the ellipses and circumferences.

Now that we know that general relativity predicts the existence of gravitational waves and we understand how to detect them, let us discuss their generation.

4.5 Gravitational waves generated by a periodic source

In the presence of sources, we saw that the equation that governs gravitational waves becomes

$$\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \qquad (4.19, \text{recalled})$$

in the Lorentz gauge, $\partial^\mu h_{\mu\nu} = 0$ and $\partial^\mu \partial_\mu \xi^\nu = 0$. We start considering the simplified case in which $T_{\mu\nu}$ is a periodic function with frequency Ω,

$$T_{\mu\nu} = \mathrm{Re}[S_{\mu\nu}(x^k) e^{-i\Omega t}] \,. \tag{4.28}$$

A more generic time-dependent energy-momentum tensor can always be thought of as a collection of Fourier modes of the form in Eq. (4.28). It is worth noting that Eq. (4.28) is anyway expected to model accurately many astrophysical sources such as binary systems, which appear to be roughly periodic.

Additionally, we assume that the region where $S_{\mu\nu} \neq 0$ is enclosed within a sphere of radius ϵ, small compared to the period of the movement and its corresponding wavelength, $\epsilon \ll 2\pi/\Omega$. This is the so-called *slow-motion* assumption, as translates to the fact that velocities have to be small compared to the speed of light. Most powerful gravitational wave sources will fail to meet this assumption, but it is applicable in many other scenarios.

Let us search for a solution of the form $h_{\mu\nu} = B_{\mu\nu} e^{-i\Omega t}$. From Eq. (4.19) we get

$$(\partial^k \partial_k + \Omega^2) B_{\mu\nu} = -16\pi S_{\mu\nu} \,. \tag{4.29}$$

In the region outside the source (that is, where $S_{\mu\nu}$ vanishes), this equation would be generically solved in terms of the spherical harmonics $Y_{lm}(\theta, \varphi)$, and we would decompose the solution as a sum $B_{\mu\nu} = A^{lm}_{\mu\nu} f_l(r) r^{-\frac{1}{2}} Y_{lm}(\theta, \varphi)$ and solve for $f_l(r)$. It turns out that the main contribution comes from the $l = 0$ term, in which case

$$B_{\mu\nu} = \frac{A_{\mu\nu}}{r} e^{i\Omega r} + \frac{Z_{\mu\nu}}{r} e^{-i\Omega r} \,. \tag{4.30}$$

with certain constants $A_{\mu\nu}$ and $Z_{\mu\nu}$. We set $Z_{\mu\nu} = 0$ as the second term in Eq. (4.30) corresponds to ingoing waves, and we want to focus on waves emitted from the source. Our problem now just consists on identifying $A_{\mu\nu}$ in terms of the source. For that let us integrate over the region V_ϵ which is sorrounded by the sphere of radius ϵ both sides of Eq. (4.29). The second term on the left-hand side is bounded by

$$\int_{V_\epsilon} \Omega^2 B_{\mu\nu} \mathrm{d}^3 x \le \max_{r<\epsilon} |B_{\mu\nu}| \frac{4\pi\epsilon^3}{3} \Omega^2 \,. \tag{4.31}$$

In our slow-motion approximation, where $\epsilon\Omega \ll 2\pi$, this term is negligible when compared to

$$\int_{V_\epsilon} \partial_k \partial^k B_{\mu\nu} \mathrm{d}^3 x = \oint_{S_R} n^k \partial_k B_{\mu\nu} \mathrm{d}^2 s \,, \tag{4.32}$$

by Gauss' theorem, with n^k a unit vector normal to the sphere S_R with radius $R > \epsilon$. Then,

$$\oint_{S_R} n^k \partial_k B_{\mu\nu} \mathrm{d}s = 4\pi R^2 \frac{\mathrm{d}B_{\mu\nu}}{\mathrm{d}r}\Big|_{r=R} \simeq -4\pi A_{\mu\nu} \tag{4.33}$$

where again we are making use of the approximation $\epsilon \ll 2\pi/\Omega$. Note that, then

$$\overline{h}_{\mu\nu} = \frac{4}{r}e^{i\Omega(r-t)}\int S_{\mu\nu}\mathrm{d}^3x = \frac{4}{r}e^{i\Omega r}\int T_{\mu\nu}\mathrm{d}^3x\,. \tag{4.34}$$

These correspond to the gravitationa waves generated by the source, neglecting terms of order $1/r^2$ and any term of order $1/r$ that is higher order in $\epsilon\Omega$.

Physical constraints allow us to simplify Eq. (4.34) considerably. Indeed, if we focus its the (μ, t) components, we can write[12]

$$\overline{h}^{\mu t} = \frac{4}{r}e^{i\Omega r}\int T^{\mu t}\mathrm{d}^3x = \frac{4}{r}\frac{e^{i\Omega r}}{i\Omega}\int \partial_t T^{\mu t}\mathrm{d}^3x\,. \tag{4.35}$$

Now we can use conservation law of $T^{\mu\nu}$, which tells us that $\partial_t T^{\mu t} = -\partial_k T^{\mu k}$, so that

$$\overline{h}^{\mu t} = -\frac{4}{r}\frac{e^{i\Omega r}}{i\Omega}\int \partial_k T^{\mu k}\mathrm{d}^3x = -\frac{4}{r}\frac{e^{i\Omega r}}{i\Omega}\oint_{S_R}\partial_k T^{\mu k}n_k\mathrm{d}^2s = 0\,, \tag{4.36}$$

where in the last equality we used Gauss' theorem and the fact the the source is localised. In conclusion, $\overline{h}_{\mu t} = 0$.

Thus, we focus in the space components of $\overline{h}$. These can be further simplified using the expression

$$\frac{\partial^2}{\partial t^2}\int T^{tt}x^l x^m\mathrm{d}^3x = 2\int T^{ml}\mathrm{d}^3x\,, \tag{4.37}$$

which is obtained using conservation of the energy momentum tensor and Gauss' theorem two times. In Eq. (4.37) we recognise the quadrupole moment of our mass distribution,

$$I^{lm} := \int T^{tt}x^l x^m\mathrm{d}^3x\,. \tag{4.38}$$

Note that, in the simplified case that we are considering, we could write $I^{lm} = D^{lm}(x)e^{-i\Omega t}$. With this we can express the non-vanishing components of Eq. (4.34) as

$$\overline{h}_{jk} = -\frac{2\Omega^2}{r}I_{jk}e^{i\Omega r}\,. \tag{4.39}$$

Finally, let us mention that it is possible to work in an equivalent to the TT gauge in this case. If we choose the axes so that at the point that we measure the wave is travelling in the z direction, Eq. (4.39) can be written as

$$\begin{aligned}
\overline{h}^{\mathrm{TT}}_{zi} &= 0\,,\\
\overline{h}^{\mathrm{TT}}_{xx} &= -\overline{h}^{\mathrm{TT}}_{yy} = -\frac{\Omega^2}{r}(\mathbf{I}_{xx} - \mathbf{I}_{yy})e^{i\Omega r}\\
\overline{h}^{\mathrm{TT}}_{xy} &= -\frac{2\Omega^2}{r}\mathbf{I}_{xy}e^{i\Omega r}\,.
\end{aligned} \tag{4.40}$$

where $\mathbf{I}_{lm} = I_{lm} - \delta_{lm}I^k_k$ is the reduced quadrupole moment tensor.

[12]Neddless to be said, we are considering the case $\Omega \neq 0$.

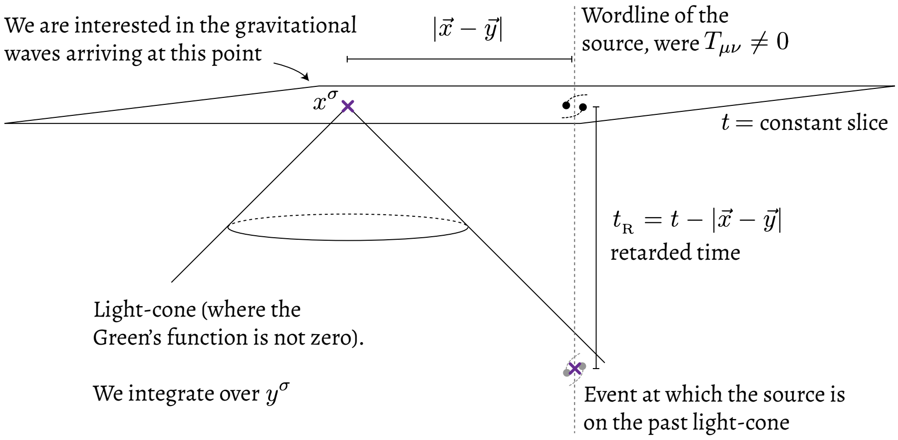

Figure 15: Sketch of the region of integration corresponding to the retarded Green's function. Even though we are only drawing one source for the sake of simplicity, the final signal received at x^σ would contain contributions from any other source on the light cone.

4.6 Gravitational waves generated by a non-periodic source

In the previous section we examined the gravitational waves that periodic sources such as binary systems of stars produce. Now, we wish to generalise the result we obtained a little bit and consider non-periodic sources. Of course, the equations that we want to solve are the same, linearised Einstein's equations in Eq. (4.19),

$$\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \qquad (4.19, \text{recalled})$$

in the Lorentz gauge, where $\partial_\mu \bar{h}^{\mu\nu} = 0$ and $\partial_\alpha \partial^\alpha \xi^\mu = 0$. Now, the energy momentum tensor can have a generic dependence on time, though we will still assume that it is non-vanishing in a small region of space. In this situation, it is useful to consider the *Green's functions*. Indeed, let G be the function which is a solution to

$$\partial_\alpha \partial^\alpha G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma)\,, \qquad (4.41)$$

where it is understood that the derivatives of ∂_α and ∂^α are taken with respect to x^σ. Then G is a Green function of our system of Eqs. (4.19). If we could find such function, the solution for $\bar{h}_{\mu\nu}$ would easily be written in terms of it,

$$\bar{h}_{\mu\nu}(x^\sigma) = -16\pi \int \mathrm{d}^4 y G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma)\,. \qquad (4.42)$$

This last statement can be checked by substituting Eq. (4.42) into Eq. (4.29).

Fortunately, mathematicians and physicists have worked a lot with this kind of problem, and we can directly borrow a solution to Eq. (4.41) from them. It is the *retarded* Green's function

$$G_\mathrm{R}(x^\sigma - y^\sigma) = -\frac{1}{4\pi|\vec{x}-\vec{y}|}\delta(|\vec{x}-\vec{y}| - (x^0 - y^0))\,\Theta(x^0 - y^0)\,. \qquad (4.43)$$

In this last expression, $\Theta(t)$ is the Heaviside step function ($\Theta(t) = 1$ if $t > 0$ and $\Theta(t) = 0$ if $t < 0$) and $\vec{\ }$ denotes the spacial components of the corresponding vector. Note that G_R is zero everywhere except on the past-light cone of the event x^σ, where it is a delta function. Thus, when substituted into Eq. (4.42),

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4\int \frac{1}{|\vec{x}-\vec{y}|} T_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y})\mathrm{d}^3 y\,, \tag{4.44}$$

it is just given the resulting gravitational wave signal arriving at $\vec{x}$ at the instant $x^0 =: t$, as a sum over all the sources on the past light-cone. Considering Eq. (4.44), and Figure 15, it makes sense to define the *retarded time* as $t_{\mathrm{R}} = t - |\vec{x}-\vec{y}|$. This is the time that it takes to the signal to get to x^σ since it is emitted at y^σ.

Given that the source now has a general dependence on time, it will be composed by several frequencies. For this reason, it is convenient to Fourier transform the time variable from the last expression. Then,

$$\begin{aligned} \mathcal{F}[h_{\mu\nu}](\omega,\vec{x}) &= \frac{4}{\sqrt{2\pi}}\int e^{i\omega t}\frac{1}{|\vec{x}-\vec{y}|} T_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y})\ \mathrm{d}t\mathrm{d}^3\vec{y} \\ &= \frac{4}{\sqrt{2\pi}}\int e^{i\omega(t-|\vec{x}-\vec{y}|)}\frac{e^{i\omega|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} T_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y})\ \mathrm{d}t\mathrm{d}^3\vec{y} \\ &= 4\int \frac{e^{i\omega|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\mathcal{F}[T_{\mu\nu}](\omega,\vec{y})\ \mathrm{d}^3\vec{y}\,, \end{aligned} \tag{4.45}$$

where to go from the first to the second line we have just multiplied by unity in the form $1 = e^{i\omega(-|\vec{x}-\vec{y}|)}e^{i\omega(|\vec{x}-\vec{y}|)}$; and to go from the second to the third we identified the Fourier transform of $T_{\mu\nu}$. When the source is far away (that is to say, its characteristic size ϵ is way smaller that the distance $|\vec{x}-\vec{y}|$ to the place from where we are measuring), we can consider $|\vec{x}-\vec{y}| = r = \text{constant}$ in the region where $T_{\mu\nu}$ does not vanish. When this approximation applies, we can extract the factor taht depends on $|\vec{x}-\vec{y}|$ from the integral in Eq.(4.45) and get

$$\mathcal{F}[h_{\mu\nu}](\omega,\vec{x}) = 4\frac{e^{i\omega r}}{r}\int \mathcal{F}[T_{\mu\nu}](\omega,\vec{y})\,\mathrm{d}^3\vec{y}\,. \tag{4.46}$$

It can be illuminating to compare this last expression with (4.34). As in the previous section, this expression can be simplified further. On the one hand, the first gauge fixing condition tells us that $\mathcal{F}[\bar{h}^{t\nu}] = i\partial_k\mathcal{F}[\bar{h}^{k\nu}]/\omega$, and so we only have to care about the spacial components of $\bar{h}_{\mu\nu}$. The equivalent of Eq. (4.34) in Fourier space allows us to write

$$\mathcal{F}[h_{ij}](\omega,\vec{x}) = \frac{4}{r}e^{i\omega r}\left(-\frac{\omega^2}{2}\int y^i y^j \mathcal{F}[T^{00}](\omega)\mathrm{d}^3\vec{y}\right) = -\frac{2e^{i\omega r}}{r}\omega^2\mathcal{F}[I_{ij}](\omega)\,. \tag{4.47}$$

Finally, transforming back to the time coordinate we obtain that

$$\bar{h}_{ij} = \frac{2}{r}\ddot{I}_{ij}(t_{\mathrm{R}})\,. \tag{4.48}$$

The fact that there is no factor $e^{i\omega r}$ in this last expression is not a "typo": it is used in the inverse Fourier transform to express the result as a function of retarded time $t_{\mathrm{R}} = t - r$. In terms of the reduced quadrupole moment tensor Eq. (4.48) becomes, in the TT gauge

$$\bar{h}^{\mathrm{TT}}_{xx} = \frac{1}{r}\left(\ddot{\mathbf{I}}_{xx}(t_{\mathrm{R}}) - \ddot{\mathbf{I}}_{yy}(t_{\mathrm{R}})\right)\,, \qquad \bar{h}^{\mathrm{TT}}_{xy} = \frac{2}{r}\ddot{\mathbf{I}}_{xy}(t_{\mathrm{R}})\,, \qquad \bar{h}^{\mathrm{TT}}_{zk} = 0\,. \tag{4.49}$$

4.7 Final remarks

One final question that we will not have time to discuss is the amount of energy carried away by gravitational waves. The result is the *gravitational Larmor formula*: the power radiated away is given by

$$P = \frac{G}{5c^5} \langle \dddot{\mathbf{I}}_{ij} \, \dddot{\mathbf{I}}^{ij} \rangle \,, \tag{4.50}$$

where dots stand for derivative with respect to time and $\langle \; \rangle$ stands for averaging over one period[13]. This expression was checked experimentally in the pulsar PSR1913+16, a binary system of two ~ 1.4 solar mass stars, separated 10^5km. The period of the orbit is ~ 8 hours. Due to energy loss, this period is reduced in 3×10^{-12} s every second. This reduction has been observed and constituted the first (indirect) evidence for the existence of gravitational waves. Hulse and Taylor, who found the pulsar in 1974, were awarded the Novel prize in physics in 1993 for their discovery.

[13]Note that we are assuming again periodic motion, even though several frequencies could be involved.

5 Advanced topics

5.1 An action for gravity

In classical physics, we like to model our systems starting with an action from which the dynamics are derived. Normally, we assume that we have a Lagrangian density $\mathcal{L}$ that depends on a set of fields ϕ_i and their derivatives, $\partial_\mu \phi_i$. Then the action is nothing but

$$S = \int \mathcal{L}(\{\phi_i, \partial_\mu \phi_i\})\, \mathrm{d}^4 x\,. \tag{5.1}$$

In this case, the equations of motions are the Euler–Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = 0\,. \tag{5.2}$$

obtained by demanding that the variation of the action S with respect to the fields ϕ_i is zero, $\delta S/\delta\phi_i = 0$. Similarly, we would like to find an action from which Einstein's equations can be obtained.

In this section, we work in natural units $\hbar = c = 1$, rather than geometrised (we keep G explicit).

5.1.1 The Einstein–Hilbert action

The dynamics in general relativity are governed by Einstein's equations, which are the equations of motion for the components of the metric. In this section, we will see that these equations are obtained by varying the *Einstein–Hilbert action*

$$S_{\text{EH}} = \frac{1}{16\pi G} \int_M \sqrt{-g} R\, \mathrm{d}^4 x \tag{5.3}$$

with respect to the metric components. In Eq. (5.3), we denoted by g the determinant of the metric $g_{\mu\nu}$, and R is its Ricci scalar. The integral is performed over the whole manifold M. One particular property of Eq. (5.3) is that it contains second derivatives of the fields with respect to which we want to vary (i.e. $g_{\mu\nu}$). We will see that this have some important consequences.

The computation is slightly easier if we vary with respect to the inverse metric $\delta g^{\mu\nu}$, rather than $\delta g_{\mu\nu}$. These are of course equivalent, as the number of degrees of freedom is the same. Note that, because $g_{\mu\nu} g^{\nu\sigma} = \delta_\mu^{\ \sigma}$, the variation of the metric and its inverse are related through

$$\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}\,. \tag{5.4}$$

The variation of S_{EH} will contain two pieces,

$$\delta S_{\text{EH}} = \frac{1}{16\pi G} \int_M \Big[\, \delta(\sqrt{-g}) R \;+\; \delta(R)\sqrt{-g}\, \Big]\, \mathrm{d}^4 x\,. \tag{5.5}$$

Taking into account that the derivative of a matrix A with respect to the components of its inverse A^{mn} is $\partial(\det A)/\partial A^{mn} = -(\det A) A_{nm} = -(\det A)(A_{(mn)} - A_{[mn]})$, in the first term we will find

$$\delta(\sqrt{-g}) = -\sqrt{-g}\, \frac{1}{2} g_{\mu\nu}\, \delta g^{\mu\nu}\,. \tag{5.6}$$

The variation of the Ricci scalar is actually a little bit more involved. Since $R = g^{\mu\nu}R_{\mu\nu}$, we will have

$$\delta(R) = R_{\mu\nu}\delta g^{\mu\nu} + \delta(R_{\mu\nu})g^{\mu\nu}\,. \tag{5.7}$$

And now the variation of the Ricci tensor gets technical. First, we have to use the *Palatini identity*, which tells us that[14] $\delta R^{\mu}{}_{\nu\rho\sigma} = 2\nabla_{[\rho}\delta\Gamma^{\mu}_{\sigma]\nu}$. In particular, $\delta(R_{\mu\nu}) = -2\nabla_{[\nu}\delta\Gamma^{\lambda}_{\lambda]\mu}$. Next, we should make use of the fact that[15]

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}\left(\nabla_\mu g_{\lambda\nu} + \nabla_\nu g_{\lambda\mu} - \nabla_\lambda g_{\mu\nu}\right)\,. \tag{5.8}$$

Taking this into account, we can finally write

$$\delta(R_{\mu\nu})g^{\mu\nu} = \nabla_\mu(g_{\rho\sigma}\nabla^\mu\delta g^{\sigma\rho} - \nabla_\nu\delta g^{\mu\nu}) =: \nabla_\mu A^\mu\,, \tag{5.9}$$

where the last equality defines A. Substituting Eqs. (5.6), (5.7) and (5.9) into Eq. (5.5) we obtain

$$\delta S_{\rm EH} = \frac{1}{16\pi G}\int_M \sqrt{-g}\Big[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \nabla_\mu A^\mu\Big]\,\mathrm{d}^4x\,. \tag{5.10}$$

In this last expression, we recognise the Einstein tensor and, using Gauss' theorem, a boundary term

$$\int_M \sqrt{-g}\,\nabla_\mu A^\mu \mathrm{d}^4x = \int_{\partial M}\sqrt{-h}\,n_\mu A^\mu \mathrm{d}^3x \tag{5.11}$$

At this point, one is tempted to claim victory, as one could argue that the boundary term should be ignored. Even though this is not the correct thing to do, let us assume that it is and fix the "mistake" later. Forgetting about the boundary term, the variation of the action $\delta S_{\rm EH}/\delta g^{\mu\nu}$ leads to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0\,, \tag{5.12}$$

which are Einstein's equations in vacuum.

How do we get Einstein's equations in the presence of matter? If the theory that we want to study also contains matter fields, the action that we will be interested in will have and extrapiece $S_{\rm m}$ containing the dynamics of matter,

$$S = S_{\rm EH} + S_{\rm m}\,. \tag{5.13}$$

For this to be consistent, we need $S_{\rm m}$ to be invariant under diffeomorphisms. Now, when S is varied with respect to $\delta g^{\mu\nu}$, the second piece gives raise to the energy momentum tensor of the corresponding matter fields

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta g^{\mu\nu}}\,, \tag{5.14}$$

[14] One may find it strange that we are writing the covariant derivative of $\delta\Gamma$. This is, however, fine: while Christoffel symbols are generically not tensors, differences of them are; and $\delta\Gamma^{\rho}_{\lambda\mu} = \Gamma^{\rho}_{\lambda\mu} - \tilde{\Gamma}^{\rho}_{\lambda\mu}$ can be understood as the difference between the connections of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$.

[15] This can be proved either by brute force (doing the derivative with respect to the metric in the formula for $\Gamma^{\rho}_{\mu\nu}$), or working in flat coordinates and realising at the end of the computation that the result does not depend on the choice of coordinates.

and Einstein's equations as in Eqs. (1.1) are recovered;

$$\frac{\delta S_{\text{EH}}}{\delta g^{\mu\nu}} = \frac{\sqrt{-g}}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 8\pi G T_{\mu\nu}\right) = 0\,. \tag{5.15}$$

Moreover, note that the energy-momentum tensor here is symmetric and satisfies $\nabla_\mu T^{\mu\nu} = 0$ by construction. There is an interesting discussion on the relation between this energy-momentum tensor in Eq. (5.14) and the one we would obtain from Noether's theorem in this webpage.

Finally, having obtained the equations of motion from S_{EH} also helps us to understand what goes wrong with general relativity near singularities. Indeed, if we asked for the most general action that respects the symmetries of the theory (i.e. it must be invariant under diffeomorphisms), we would write

$$\tilde{S} = \frac{1}{16\pi G}\int \sqrt{-g}\left(-2\Lambda + R + c_1 R^2 + c_2 R^{\mu\nu}R_{\mu\nu} + c_3 R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho} + \cdots\right)\mathrm{d}^4x\,. \tag{5.16}$$

The first term would just lead to the presence of the cosmological constant Λ in Einstein's equations, while the second is precisely that in the Einstein–Hilbert action. The crucial observation now is that curvature quantities are second derivatives of the metric, $R \sim \partial^2 g$, and thus have dimensions of $(\text{length})^{-2}$. This means that the coefficients c_i will be suppressed by some powers of the Newton's constant[16] G, as it is the unique dimensionful quantity in the theory. This justifies ignoring all those terms whenever curvatures are small, but one would certainly want to include them in highly curved regions. At the same time, when curvatures are large the theory of general relativity should be modified to include those terms. How to do so consistently is not completely understood and beyond the scope of these lectures.

Before changing topic, we still have to understand the boundary term Eq. (5.11) that we neglected in our derivation. In the next section we will discuss how to treat it properly.

5.1.2 The Gibbons–Hawking–York boundary term

In the previous section, we got Einstein's equations from the Einstein–Hilbert action in Eq. (5.3). In our derivation, we ignored the boundary term written in Eq. (5.11). We should not blame ourselves too much for that; the subtlety was not taken into consideration until Gibbons and Hawking pointed it out in [15]. That was more than sixty years after Hilbert proposed Eq. (5.3), so people could get along with it.

To understand what we did incorrectly, let us consider a simpler case. We want to argue that the problem arises from R containing second derivatives of the metric. A similar problem would appear if we were to take the Lagrangian

$$I_1 = -\frac{1}{2}\int_{t_0}^{t_1} \mathrm{d}t q\ddot{q} \tag{5.17}$$

to describe a free one-dimensional classical particle, instead of the standard one Lagrangian $I_2 = \int_{t_0}^{t_1}\mathrm{d}t\frac{1}{2}\dot{q}^2$, build just from the kinetic energy. We expect the equations of motion of

[16] Newton's constant G has dimensions of $(\text{length})^2$.

this system to be $\ddot{q} = 0$. If we vary I_1 with respect to δq, we get

$$\delta I_1 = -\int_{t_0}^{t_1} \mathrm{d}t\, \ddot{q}\delta q + \frac{1}{2}\int_{t_0}^{t_1} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\left(-q\delta\dot{q} + \dot{q}\delta q\right) = -\int_{t_0}^{t_1} \ddot{q}\delta q\, \mathrm{d}t + \frac{1}{2}\Big[-q\delta\dot{q} + \dot{q}\delta q\Big]_{t_0}^{t_1}. \quad (5.18)$$

Crucially, we demand that δq vanishes at the boundary (i.e. the end points t_0 and t_1). However, further demanding $\delta\dot{q} = 0$ at the boundary would correspond to additional conditions that we are not allowed to require. Thus, if we want something like the integral in Eq. (5.19) to give rise to the equations of motion that we want (i.e. $\ddot{q} = 0$), we need to modify it by subtracting the boundary term,

$$\tilde{I}_1 = I_1 + \frac{1}{2}q\dot{q}\Big|_{t_0}^{t_1} = \frac{1}{2}\int \dot{q}^2\, \mathrm{d}t\ = I_2\,. \quad (5.19)$$

Note that with the extra boundary term we get rid of the second derivatives. Actually, we recovered I_2.

With this simpler example in mind, it is easier to understand the most complicated case of the Einstein–Hilbert action. Recall that the integrand of the boundary term we were left with was

$$n^\mu A_\mu = n^\mu g^{\nu\rho}\left(\nabla_\rho \delta g_{\mu\nu} - \nabla_\mu \delta g_{\nu\rho}\right) = \left(n^\mu h^{\nu\rho} \mp n^\mu n^\nu n^\rho\right)\left(\nabla_\rho \delta g_{\mu\nu} - \nabla_\mu \delta g_{\nu\rho}\right), \quad (5.20)$$

where in the last equality we used the definition of the induced metric $h_{\mu\nu} = g_{\mu\nu} \pm n_\mu n_\nu$ of the boundary ∂M, Eq. (2.25). The choice of upper or lower indices depends on whether ∂M is spacelike or timelike, respectively.

If we now inspect Eq. (5.20), we realise that the second term vanishes, for $n^\mu n^\nu n^\rho$ is symmetric under the exchange of ρ and μ while the parenthesis is antisymmetric. Moreover, the term $n^\mu h^{\nu\rho}\nabla_\rho \delta g_{\mu\nu} = 0$ is also zero, as it is the tangential derivative of the variation of the metric, and $\delta g^{\mu\nu}|_{\partial M} = 0$ (the metric is not allowed to vary on the boundary). Then,

$$n^\mu A_\mu = -n^\mu h^{\nu\rho}\nabla_\mu \delta g_{\nu\rho}\,. \quad (5.21)$$

Interestingly, we can relate this with the variation of the extrinsic curvature on the boundary. Indeed, recall from Eq. (2.26) that $K = h^{\mu\nu}K_{\mu\nu} = h^\rho_\mu \nabla_\rho n^\mu = h^\rho_\mu(\partial_\rho n^\mu + \Gamma^\mu_{\rho\alpha}n^\alpha)$. Then, taking into account that $\delta h^\rho_\mu = 0$ if $\delta g^{\mu\nu}|_{\partial M} = 0$,

$$\delta K = n^\alpha h^\rho_\mu \delta\Gamma^\mu_{\rho\alpha} = \frac{1}{2}n^\alpha h^{\mu\nu}\nabla_\alpha \delta g_{\mu\nu}\,. \quad (5.22)$$

In the last equality we used Eq. (5.8). Now, comparing our last two expressions we conclude that our boundary term is $n^\mu A_\mu = -2\delta K$. Consequently, an extra piece to cancel this boundary term needs to be added to S_{EH} if we want an action from which Einstein's equations are obtained. This piece, that we denote by S_{GHY}, is called the *Gibbons–Hawking–York* term and the final action for gravity becomes

$$S_{\text{G}} = S_{\text{EH}} + S_{\text{GHY}} = \frac{1}{16\pi G}\int_M \sqrt{-g}R + \frac{1}{8\pi G}\int_{\partial M}\sqrt{-h}K\,. \quad (5.23)$$

This new term is important to be taken into consideration whenever the boundary plays an important role, such as in holography. The term was introduced in [15] because they were studying partition functions in quantum gravity, and for that they wanted to have an action which depended only on the first derivatives of the metric.

5.2 Black hole thermodynamics

5.2.1 The Penrose process

When we think of black holes, the first thing that comes to our mind is the statement "nothing can escape". Indeed, that is true, in the sense that nothing that crosses the horizon will ever be able to cross it back. For this reason, it was a surprise when Penrose showed in 1971 that it is possible to extract energy from a rotating black hole [16]. The key observation is that in the ergosphere of a Kerr black hole, whose metric was given in Eq. (3.74), the Killing vector corresponding to time translations at infinity, $\xi = \partial/\partial t$, becomes spacelike. Consequently, particles in the ergosphere can have negative energy, as we shall see.

Imagine that we throw from far away a particle with momentum P into the black hole. The energy of the particle is $E_0 = -\xi^\mu P_\mu = -P_t > 0$ as in Eq. (3.79). This is positive because static observers at infinity have ξ as their proper time, and need to see the particle inside their lightcone.

Now, the particle will split into two different particles with momenta Q and T when it is inside the ergosphere. Conservation of total momentum requires $P = Q + T$ when they split. We will now take advantage of the absence of static observers inside the ergosphere to claim that one of the particles can have negative energy. Indeed, in the ergoregion we find stationary observers, rather than static ones. As we discussed around Eq. (3.85), they are rotating with a certain constant angular velocity $\Omega \in (\Omega_{\min}, \Omega_{\max})$ with respect to static observers at infinity. The quadri-velocity of stationary observers was

$$U = U^t \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) , \qquad \text{(3.83, recalled)}$$

This becomes $U = U^t \chi = U^t(\partial/\partial t + \Omega_{\mathrm{H}} \partial/\partial\varphi)$ as we approach the horizon, with $\Omega_{\mathrm{H}} = a/(r_+^2 + a^2)$ the angular velocity of the horizon given in Eq. (3.82). Non-superluminal propagation in this case implies

$$0 < -\chi_\mu Q^\mu = -Q_t - \Omega_{\mathrm{H}} Q_\varphi = E_1 - \Omega_{\mathrm{H}} L_1 \,. \qquad (5.24)$$

In particular, the energy of one of the particles can be negative, as long as $E_1 > \Omega_{\mathrm{H}} L_1$. If so, note that $L_1 < 0$ is also negative, and so the particle with negative energy must rotate in the opposite direction that the black hole. The second particle, will have energy $E_2 = E_0 + |E_1|$. Remarkably, there are particular choices of E_1 and L_1 (like precisely the one Penrose discusses in [16]) for which this second particle escapes to infinity while the first one falls into the black hole, effectively carrying energy out of the black hole. After the process, we will be left with a rotating black hole with mass $\tilde{M} = M + \delta M$ and angular spin $\tilde{J} = J + \delta J$, where $\delta M = E_1$ and $\delta J = L_1$, both of which are negative in the present case. The interpretation is that we are extracting rotational energy out of the Kerr black hole.

How much energy can we extract if we repeat the process many times? Clearly, it will not be all of it: in the process, the black hole is spinning down, and will eventually become a Schwarschild black hole, from which no more energy can be extracted. To make this quantitative, let us investigate how the area of the black hole changes in the process. The area of the surface $t =$ constant, $r = r_+ = G(M + M\sqrt{1 - J^2/M^2})$ is

$$A_{\rm H} = 4\pi(r_+^2 + a^2) = 8\pi G^2(M^2 + \sqrt{M^4 - J^2})\,. \tag{5.25}$$

Remember that $a = JG/M$. From this, we get that

$$\delta A_{\rm H} = \frac{8\pi G}{\kappa}\left(\delta M - \Omega_{\rm H}\delta J\right)\,, \tag{5.26}$$

where $\kappa = G\sqrt{M^2 - a^2}/(r_+^2 + a^2)$ whose physical meaning we will discuss in the next section. Remarkably, the condition in Eq. (5.24) tells us that the area can only grow, $\delta A_{\rm H} \geq 0$. This also allows us to provide a bound on the amount of energy that we can extract. Indeed, if we define

$$M_{\rm I}^2 = \frac{1}{2}\left(M^2 + \sqrt{M^4 - J^2}\right)\,, \tag{5.27}$$

where the prefactor is fixed so that $M_{\rm I}$ corresponds to the mass of the balck hole when it does not rotate. Moreover, Eq. (5.26) also tells us that $\delta M_{\rm I} \geq 0$. Therefore, the maximum mass that we can extract is

$$\Delta M = M - M_{\rm I} = M\left(1 - \frac{1}{\sqrt{2}}\left(1 + \sqrt{1 - J^2/M^4}\right)\right)\,. \tag{5.28}$$

In particular, from a extremal black hole (i.e. $J = M^2$) on could in principle extract $\Delta M/M = 1 - 1/\sqrt{2} \simeq 30\%$ of its energy by virtue of the Penrose process.

5.2.2 Area theorem

In the previous section, we saw that Penrose process cannot reduce the area of a rotating black hole. This is a particular case of a more general results that Stephen Hawking proved in 1971: total area of black holes cannot decrease in [17]. We will not prove it here, but comment on it.

The theorem assumes a rather generic situation. Namely, that the *null energy condition* holds: for any timelike vector field K, it is true that $T_{\mu\nu}K^\mu K^\nu \geq 0$ or, equivalently, $R_{\mu\nu}K^\mu K^\nu \geq 0$. It also assumes that the spacetime is strongly asymptotically predictable, which is to say that there are no naked singularities. From these assumptions, it follows that *the total area of future event horizon cannot decrease*,

$$\Delta A \geq 0\,. \tag{5.29}$$

In particular, black holes cannot split.

This theorem sets a bound on the amount of energy that can be radiated away in the form of gravitational waves. Imagine that we start with two Schwarschild black holes with area

$$A_1 = 4\pi(2GM_1)^2\,, \qquad A_2 = 4\pi(2GM_2)^2\,. \tag{5.30}$$

Generically, these black holes will approach each other and collide. If the initial black holes were both initially at rest (with respect to an observer at infinity), the merger will take place via a head-on collision. Thus, spherical symmetry will be recovered when the system settles down through the emission of gravitational waves. Consequently, we will end up with a final spherical black hole with area $A_f = 4\pi(2GM_f)^2$. The area theorem states that

$$A_1 + A_2 \leq A_f\,, \tag{5.31}$$

which means that

$$M_1^2 + M_2^2 \leq M_f^2 = (M_1 + M_2 - E_{\rm GW})^2\,, \tag{5.32}$$

with $E_{\rm GW}$ the amount of energy radiated away in the form of gravitational waves. In particular,

$$E_{\rm GW} \leq M_1 + M_2 - \sqrt{M_1^2 + M_2^2}\,. \tag{5.33}$$

This results states, for example, that for two equally massive spherical black holes gravitational waves could carry, at most, $1 - 1/\sqrt{2} \simeq 30\%$ of the total mass of the system. Simulations of black hole collisions show that it is actually way less efficient than that, with only between 1% and 2% being radiated away. In the first detected collision of gravitational waves [11], the initial black hole masses were 36 and 29 solar masses and 3 solar masses ($\sim 5\%$) were radiated away.

5.2.3 Surface gravity and the four laws of black hole mechanics

Let us imagine for a second that we are a static observer near the horizon of a Schwarschild black hole. Expanding the metric about $r = 2GM$, using $r = 2GM + \epsilon^2 + \cdots$, we get $\mathrm{d}r^2 = 4\epsilon^2\mathrm{d}\epsilon^2 + \cdots$ and the metric becomes, approximately

$$\mathrm{d}s^2 \simeq -\frac{\epsilon^2}{2GM}\mathrm{d}t^2 + 8GM\mathrm{d}\epsilon^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2\,. \tag{5.34}$$

We have used that $r^2\mathrm{d}\Omega_2 \simeq \mathrm{d}Y^2 + \mathrm{d}Z^2$ looks nearly flat close to the horizon ($\epsilon \ll \sqrt{GM}$). Now, performing a change of variables $\bar{x} = \epsilon\sqrt{8GM}$ we arrive to

$$\mathrm{d}s^2 \simeq -\frac{1}{16GM}\bar{x}^2\mathrm{d}t^2 + \mathrm{d}\bar{x}^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2\,, \tag{5.35}$$

which is the metric of the Rindler space. This is nothing but Minkowski space as sensed by observers with constant acceleration κ given by

$$\kappa = \frac{1}{4GM}\,, \tag{5.36}$$

Note that this is the acceleration that the engine of our spacecraft should achieve to keep us at rest hovering over the black hole. This is the reason why we will refer to κ as *surface gravity*. The fact that such observers appear to be accelerating in Minkowski space is a consequence of the equivalence principle.

Surface gravity can be defined for any black hole solution. However, we have already argued that in some cases, such as for the Kerr black hole, static observers do not exist near the horizon. For this reason, we need to be a little bit more precise about what we mean by surface gravity.

Horizons are boundaries of regions from which nothing, not even light, can escape. Mathematically, this is encoded by the fact that there exists a vector field χ^μ that becomes null on the horizon $\chi_\mu \chi^\mu = 0$; and that is hypersurface normal, $\chi_{[\mu} \nabla_\nu \chi_{\rho]} = 0$. This vector field defines κ through

$$\nabla^\mu (\chi_\nu \chi^\nu) = -2\kappa \chi^\mu \tag{5.37}$$

evaluated at the horizon. For the Kerr black hole, $\chi = \partial/\partial t + \Omega_{\rm H} \partial/\partial \varphi$ would be the vector field with the desired properties, and surface gravity becomes $\kappa = G\sqrt{M^2 - a^2}/(r_\pm^2 + a^2)$. Notice that this is precisely the constant that appeared in Eq. (5.26). Let us emphasise that κ is a constant on the horizon. Actually, it is possible to prove that κ, defined by Eq. (5.37) evaluated on a horizon of any stationary black hole solution is always constant.

The results obtained so far fostered Bardeen, Carter and Hawking [18] to point out an interesting analogy between black hole mechanics and thermodynamic systems, summarised in the following table:

	Thermodynamics	**Black hole mechanics**
$0^{\rm th}$ law	T is uniform on equilibrium	κ is uniform on a Killing horizon
$1^{\rm st}$ law	$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V$	$\delta M = \dfrac{\kappa}{8\pi G}\delta A + \Omega_{\rm H} \delta J$
$2^{\rm nd}$ law	$\Delta S \geq 0$	$\Delta A \geq 0$
$3^{\rm rd}$ law	$T = 0$ cannot be achieved in a finite number of steps	$\kappa = 0$ cannot be achieved in a finite number of steps

The zeroth law of black hole mechanics corresponds to what we have just discussed. On the other hand, the first law is obtained from Eq. (5.26), isolating δM. Even though it appeared in the context of the Penrose process, in [18] it was proven to hold with more generality. Moreover, the second law is just the area theorem explained in the previous section. Finally, for the third law just a plausibility argument was given, noticing that the $\kappa = 0$ in the Kerr solution is achieved in the extremal case $J = M^2$, which would require infinite divisibility of matter and infinite time to be achieved.

In [18] this was taken just as an analogy. Actually, it was argued that the temperature of a black hole is always strictly zero, for when put in a thermal contact (say, with a radiating body), they would just absorb any sort of radiation. Therefore, it seemed they would never reach thermal equilibrium. Similarly, they argued that there appear to be mechanisms by which the entropy of a black hole is increased while its area is kept constant.

By the same time, Jacob Bekenstein examined those arguments regarding the relation between area and entropy, and claimed that one could indeed make sense of interpreting the area of the black hole as some sort of entropy [19, 20]. He rephrased the second law of thermodynamics as *"when common entropy goes down a black hole, the common entropy in the black hole exterior plus the black hole entropy never decreases"*,

$$\Delta S_{\rm ext} + \Delta S_{\rm BH} \geq 0\,, \tag{5.38}$$

with $S_{\rm BH} \propto A$ and $S_{\rm ext}$ is the entropy contained in the universe outside black holes.

Famously, Hawking changed his opinion when he proved that black holes do indeed radiate. With that, he managed to provide the correct proportionality factor between the entropy and the area; and between temperature and surface gravity:

$$S_{\rm BH} = \frac{A}{4G}\,, \qquad T_{\rm H} = \frac{\kappa}{2\pi}\,. \tag{5.39}$$

Next, we would like to argue that Eqs. (5.39) indeed give the entropy and temperature of stationary black holes. Hawking proved it in [21] by means of studying quantum field theory in curved spacetime. Understanding Hawking's approach is beyond the scope of these lectures. Instead, we will first learn how quantum field theory is done at finite temperature and show how Eqs. (5.39) are deduced from it.

5.2.4 Quantum field theory at finite temperature

In this section we discuss quantum field theory at finite temperature. First, we need to recall some concepts from Statistical Physics.

- **Boltzmann factor and the partition function.** In a thermal equilibrium configuration, the probability p_i of a given state with energy E_i is proportional to the Boltzmann factor,

$$p_i = \frac{1}{\mathcal{Z}} e^{-\beta E_i}\,. \tag{5.40}$$

Here, $\mathcal{Z}$ is the *partition function*, which is just the normalization factor that ensures that all the probabilities sum up to unity,

$$\mathcal{Z} = \sum_i e^{-\beta E_i}\,. \tag{5.41}$$

Moreover, $\beta = (k_{\rm B} T)^{-1}$ and we will be setting the Boltmann constant to unity, $k_{\rm B} = 1$. Note that Eq. (5.41) assumes a discrete spectrum of energies, but this can be easily generalised to the continuum case using integrals. Actually, if the system is quantum, we can write the partition function as

$$\mathcal{Z} = \sum_i \langle i | e^{-\beta \hat{H}} | i \rangle = \int {\rm d}x \langle x | e^{-\beta \hat{H}} | x \rangle\,, \tag{5.42}$$

where $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ is the Hamiltonian of the system. In the last step we have just changed basis to position space. After some manipulations (see Appendix A), this can be written as

$$\mathcal{Z} = \int \mathcal{D}x \exp \int_0^\beta {\rm d}\tau L_E\,, \tag{5.43}$$

with $L_E = \frac{m}{2}(\partial x/\partial\tau)^2 + V(x(\tau))$ and $x(0) = x(\beta)$. With $\int \mathcal{D}x$, the integral over all possible periodic configurations $x(\tau)$ is meant. By periodic, we mean that $x(0) = x(\beta)$. Remarkably, this is can be obtained from the path integral of the theory

$$Z = \int \mathcal{D}x \,\exp \int_{-\infty}^{\infty} {\rm d}t L\,, \qquad L = \frac{m}{2}\left(\frac{\partial x}{\partial t}\right)^2 - V(x)\,; \tag{5.44}$$

after analytic continuation to imaginary time, $t \to -i\tau$ and declaring τ to be compact and $x(\tau)$ periodic.

- **Generalisation to quantum field theories.** We have just seen that we can obtain the partition function from the path integral of a theory. Actually, the same is true in quantum field theories: given a theory whose Lagrangian density is $\mathcal{L}$, its path integral reads

$$Z = \int \mathcal{D}\phi \exp\left[i \int \mathrm{d}^4 x \, \mathcal{L}\right] . \tag{5.45}$$

Then, the finite-temperature partition function can be obtained just by Wick rotating the time direction $t \to -i\tau$,

$$\mathcal{Z} = \int \mathcal{D}\phi \exp\left[- \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\vec{x} \, \mathcal{L}_E\right] , \tag{5.46}$$

and demanding periodicity[17] of the fields, $\phi(0, \vec{x}) = \phi(\beta, \vec{x})$. There is a discussion on how periodicity appears in this context in Appendix B.

In conclusion, we can think of quantum field theories at finite temperature as being defined in Minkowski space with compactified imaginary time, $t = -i\tau$. Thus, the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \, , \tag{5.47}$$

with $\tau \in (0, \beta)$ being a periodic coordinate.

In quantum field theory, the path integral is integrated over the fields, $\mathcal{D}\phi$. If we wanted to take a similar approach to quantum gravity, we would also need to allow for different configurations of spacetime, and integrate over all possible metrics, $\mathcal{D}g$. If we want to do this at finite temperature, we would have to consider the following partition function,

$$\mathcal{Z}_{\mathrm{QG}} = \int \mathcal{D}g \exp[-S_E] \, , \tag{5.48}$$

with S_E being the Euclidean version of Eq. (5.23), with compactified time direction. This is of course complicated, and it is even difficult to understand what we mean by Eq. (5.48). However, we can take a *saddle point approximation*: we treat quantum corrections as perturbations over a classical solution,

$$g_{\mu\nu} = g_{\mu\nu}^{(\mathrm{cl})} + \delta g_{\mu\nu} + \cdots \tag{5.49}$$

We can think of $\delta g_{\mu\nu}$ as being suppressed by $\hbar$, which we have set to $\hbar = 1$. Then, expanding $\mathcal{Z}_{\mathrm{QG}}$ around the classical solution leads to

$$\mathcal{Z}_{\mathrm{QG}} \simeq \exp[-S_E[g^{(\mathrm{cl})}]] \int \mathcal{D}\delta g \exp[-\delta S_E] + \cdots \tag{5.50}$$

Clearly, the classical solution gives the dominant contribution to the path integral.

[17] Actually, it turns out that fermionic fields have to be antiperiodic.

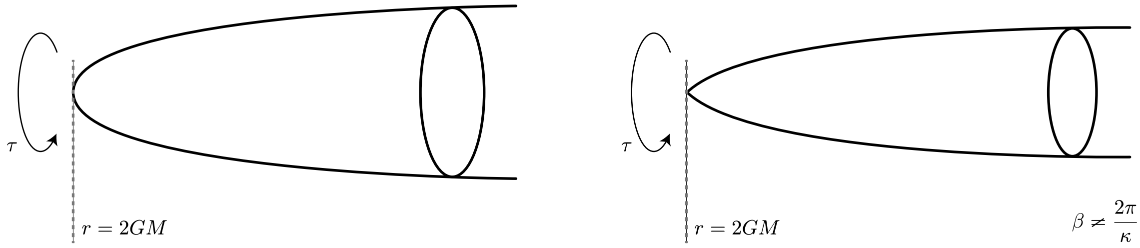

Figure 16: Behaviour of the Euclidean time circle near the horizon of a black hole. Regularity imposes that $\beta = 2\pi/\kappa$, as otherwise a conical singularity would be present, as in the right picture.

Let us apply this to the Schwarschild geometry. First, we write down the Euclidean solution, obtained by analytically continuing Eq. (3.6) to imaginary time,

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}\tau^2 + \left(1 - \frac{2GM}{r}\right) \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2 \,. \tag{5.51}$$

Expanding this metric about $r = 2GM$, we get the Euclidean version of Eq. (5.35),

$$\mathrm{d}s^2 \simeq \kappa^2 \bar{x}^2 \mathrm{d}\tau^2 + \mathrm{d}\bar{x}^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 \,, \tag{5.52}$$

where now $\tau \sim \tau + \beta$ is a periodic coordinate and $\kappa = 1/(4GM)$ is surface gravity. Remarkably, if we want the metric to be regular near the horizon, the period β needs to have a particular value. Indeed, changing coordinates in Eq. (5.52) to $\tau = \kappa^{-1}\alpha$, the metric becomes

$$\mathrm{d}s^2 \simeq \bar{x}^2 \mathrm{d}\alpha^2 + \mathrm{d}\bar{x}^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 \,. \tag{5.53}$$

In particular, the $\bar{x}$ and α directions give raise to $\mathbb{R}^2$ written in polar coordinates. But for this to be regular we need that $\alpha \in (0, 2\pi)$, for otherwise we would have a conical singularity, as depicted in Fig. 16. This requirement imposes that $\tau \in (0, 2\pi/\kappa)$ and thus

$$T_{\mathrm{H}} = \beta^{-1} = \frac{\kappa}{2\pi} \,, \tag{5.54}$$

which is Hawking temperature. The rest of the thermodynamical variables are obtained via the standard thermodynamic relations. Indeed, varyinf in mind that the partition function is related to the free energy F as $\mathcal{Z} = e^{-\beta F}$, the energy and entropy become

$$E = -\partial_\beta \log \mathcal{Z} \,, \qquad S = -\left(\beta \frac{\partial}{\partial \beta} - 1\right) \log \mathcal{Z} \,. \tag{5.55}$$

In particular, $F = E - TS$ with $T = \beta^{-1}$.

5.2.5 Aspects of Hawking radiation

Let's finish with some comments regarding Hawking radiation. Not only did Hawking find that black holes have a temperature, but he showed that they radiate and the corresponding spectrum is that of a black body. In particular, the power of emission is that of a black body

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sigma A T^4 \,, \tag{5.56}$$

with σ the Stefan-Boltzmann "constant", which depends on the field being radiated. In particular, for Schwarschild,

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\sigma}{256\pi^3}\frac{1}{M^2}\,, \tag{5.57}$$

from which we conclude that the time of evaporation is $t_{\mathrm{ev}} \propto M^3$. Including all the factors, Hawking temperature for a Schwarschild black hole becomes

$$T_{\mathrm{H}} = \frac{\hbar c^3}{8\pi k_{\mathrm{B}} g M} \simeq 6\cdot 10^{-6}\frac{M_\odot}{M}\,\mathrm{K}\,. \tag{5.58}$$

In particular, the temperature of one solar mass black hole is negligible, compared to the CMB temperature (2.7 K). Concerning the entropy,

$$S_{\mathrm{BH}} = \frac{c^3}{G\hbar}\frac{A}{4} \simeq 10^{76}\left(\frac{M}{M\odot}\right)^2\,. \tag{5.59}$$

This may not say much. But baring in mind that the entropy in the CMB is $S_{\mathrm{CMB}} \sim 10^{87}$, we conclude that most of the entropy of the Universe is in the form of black holes, as a single supermassive black hole with $M = 10^9 M_\odot$ already has $S \sim 10^{94}$ according to Eq. (5.59).

Finally, note that the specific heat is

$$c = \frac{\mathrm{d}E}{\mathrm{d}T} = \frac{\mathrm{d}M}{\mathrm{d}T} = -8\pi M^2 < 0 \tag{5.60}$$

is negative, which means that a black hole is thermodynamically unstable.

$$c = \frac{\mathrm{d}E}{\mathrm{d}t}. \tag{5.61}$$

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A Partition function for a quantum system

We want to compute

$$\mathcal{Z} = \int \mathrm{d}x \langle x|e^{-\beta\hat{H}}|x\rangle \,. \tag{A.1}$$

We will be switching between position basis and momentum basis,

$$\hat{x}|x\rangle = \hat{x}|x\rangle \,; \qquad \hat{p}|x\rangle = -i\partial_x|x\rangle \,; \tag{A.2}$$

and, of course, $[\hat{x},\hat{p}] = i$. In our convention for the normalisation of the basis,

$$1 = \int \frac{\mathrm{d}p}{2\pi}|p\rangle\langle p| \,, \qquad 1 = \int \mathrm{d}x|x\rangle\langle x| \,. \tag{A.3}$$

The first thing to note is that $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ contains several of operators. Recall that the exponentiation of two operators is such that

$$e^A e^B = \exp(A + B + \frac{1}{2}[A,B] + \cdots) \,. \tag{A.4}$$

For this reason, it is useful to split the Hamiltonian in N smaller pieces,

$$\mathcal{Z} = \int \mathrm{d}x \langle x|e^{-\beta\hat{H}}|x\rangle = \int \mathrm{d}x \langle x|e^{-\epsilon\hat{H}} \,\cdots\, e^{-\epsilon\hat{H}}|x\rangle \,, \tag{A.5}$$

with $\epsilon = \beta/N$. What we have done is fine since $[\hat{H},\hat{H}] = 0$. Next, we want to make use of

$$\begin{aligned} \langle p_n|e^{-\epsilon\hat{H}}|x_n\rangle &= \exp\left[-\epsilon\frac{p_n^2}{2m} - \epsilon V(x_n) + \mathcal{O}(\epsilon^2)\right]\langle p_n|x_n\rangle \\ &= \exp\left[-\epsilon\frac{p_n^2}{2m} - ip_n x_n - \epsilon V(x_n) + \mathcal{O}(\epsilon^2)\right] \,, \end{aligned} \tag{A.6}$$

where n is just a label. To do so, we insert $1 = (2\pi)^{-1}\int \mathrm{d}p_n|p_n\rangle\langle p_n|$ to the left of every $e^{-\epsilon\hat{H}}$ factor, and $1 = \int \mathrm{d}x_N|x_N\rangle\langle x_N|$ to its right. In this way, relabeling $x = x_{N+1}$, the partition function becomes

$$\begin{aligned} \mathcal{Z} = \int \mathrm{d}x_{N+1}\langle x_{N+1}| &\left[\int \frac{\mathrm{d}p_N}{2\pi}|p_N\rangle\langle p_N|\right] e^{-\epsilon\hat{H}} \left[\int \mathrm{d}x_N|x_N\rangle\langle x_N|\right]\cdots \\ &\cdots\left[\int \frac{\mathrm{d}p_{N-1}}{2\pi}|p_1\rangle\langle p_1|\right] e^{-\epsilon\hat{H}} \left[\int \mathrm{d}x_1|x_1\rangle\langle x_1|\right]|x_{N+1}\rangle \,. \end{aligned} \tag{A.7}$$

Note that, from the last term, we get $\langle x_1|x_{N+1}\rangle = \delta(x_1 - x_{N+1})$. Therefore, rearranging the Eq (A.7) we arrive to

$$\begin{aligned} \mathcal{Z} &= \int \prod_{i=1}^{N} \frac{\mathrm{d}x_i \mathrm{d}p_i}{2\pi}\langle x_{i+1}|p_i\rangle\langle p_i|e^{-\epsilon\hat{H}}|x_i\rangle \\ &= \int \left(\prod_{i=1}^{N} \frac{\mathrm{d}x_i \mathrm{d}p_i}{2\pi}\right) \exp\left[-\epsilon\sum_{j=1}^{N}\left(\frac{p_j^2}{2m} - ip_j\frac{x_{j+1}-x_j}{\epsilon} + V(x_j) + \mathcal{O}(\epsilon)\right)\right]_{x_{N+1}=x_1} \,, \end{aligned} \tag{A.8}$$

where we used Eq. (A.6) and the fact that $\langle x|p\rangle = e^{ipx}$. We ignored the terms $\mathcal{O}(\epsilon^2)$ because we will eventually take the limit $\epsilon \to 0$ (equivalently, $N \to \infty$).

The next observation is that Eq. (A.8) contains a set Gaussian integrals over the momentum, which can be performed and lead to

$$\mathcal{Z} = \int \left(\prod_{i=1}^{N} \frac{\mathrm{d}x_i}{(2\pi\epsilon/m)^{-\frac{1}{2}}} \right) \exp \left[-\epsilon \sum_{j=1}^{N} \left(\frac{m}{2} \left(\frac{x_{j+1} - x_j}{\epsilon} \right)^2 + V(x_j) \right) \right]_{x_{N+1}=x_1} , \tag{A.9}$$

Now we take the continuous limit $\epsilon \to 0$. The different parts in Eq. (A.9) become

$$\epsilon \sum_{j=1}^{N} \mapsto \int_0^\beta \mathrm{d}\tau \,, \qquad \frac{x_{j+1} - x_j}{\epsilon} \mapsto \frac{\partial x}{\partial \tau} \,, \qquad \left(\prod_{i=1}^{N} \frac{\mathrm{d}x_i}{(2\pi\epsilon/m)^{-\frac{1}{2}}} \right) \mapsto \int \mathcal{D}x \,. \tag{A.10}$$

These can be confusing, but let us try to understand them. If we focus in what we have inside the exponential in Eq. (A.9), after taking the continuous limit we go from a discretised set of $\{x_i\}$ to a continuous function $x(\tau)$, which we are integrating between $\tau \in (0, \beta)$. The condition $x_{N+1} = x_1$ translates to the requirement that the function is periodic, $x(0) = x(\beta)$. Of course, in the second element in Eq. (A.10) we recognise the derivative with respect to τ.

The third identification in Eq. (A.10) is perhaps the most confusing. Before taking the continuous limit, we were performing N integrals of the type $\int \mathrm{d}x_i$, which means that every of the x_i was allowed to take any value. In $\epsilon \to 0$ limit this means that we have to integrate over *all possible functions* $x(\tau)$, subjected to the condition $x(0) = x(\beta)$. In the end, the partition function becomes

$$\mathcal{Z} = \int \mathcal{D}x \, e^{-S_E} \,, \qquad S_E = \int_0^\beta \mathrm{d}\tau L_E \,, \tag{A.11}$$

where

$$L_E = \frac{m}{2} \left(\frac{\partial x}{\partial \tau} \right)^2 + V(x(\tau)) \,. \tag{A.12}$$

Note that Eq. (A.13) looks very similar to the usual, zero-temperature path integral in Minkowski space,

$$Z = \int \mathcal{D}x \, e^{iS} \,, \qquad S = \int_{-\infty}^{\infty} \mathrm{d}t L \,, \tag{A.13}$$

where L is the Lagrangian of the system,

$$L = \frac{m}{2} \left(\frac{\partial x}{\partial t} \right)^2 - V(x(t)) \,. \tag{A.14}$$

We arrive to a very important conclusion: we can obtain the finite-temperature partition function $\mathcal{Z}$ directly from the path integral Z of a given theory, by performing a Wick-rotation to imaginary time, $t \to -i\tau$, and demanding that the functions are periodic. This is true not only in quantum mechanics, but also quantum field theory.

B Technicalities about QFT at finite temperature

B.1 Density matrices and thermal states

If the system we are describing is quantum, a particular state S will be described with a density matrix ρ_S. We say that the state is pure when it is associated to a particular element $|\psi\rangle$ of the Hilbert space $\mathcal{H}$, and thus can be written as

$$\rho_{|\psi\rangle} = |\psi\rangle\langle\psi|\,. \tag{B.1}$$

Otherwise, we say that it is a *mixed state*, which we could decompose as

$$\rho_S = \sum_i p_i |i\rangle\langle i| \neq |\psi\rangle\langle\psi|\,, \qquad \forall\psi \in \mathcal{H}. \tag{B.2}$$

Here $|i\rangle$ is a basis of the Hilbert space and p_i is the probability of $|i\rangle$. Indeed, given a density matrix ρ_S and an operator $\mathcal{O}$, the expectation value of $\mathcal{O}$ is given by

$$\langle\mathcal{O}\rangle_S = \sum_i p_i\langle i|\mathcal{O}|i\rangle = \sum_j\sum_i p_i\langle i|\mathcal{O}|j\rangle\langle j|i\rangle = \sum_j\langle j|\sum_i p_i|i\rangle\langle i|\mathcal{O}|j\rangle = \mathrm{Tr}(\rho_S\mathcal{O})\,. \tag{B.3}$$

In particular, the expectation value of the projection operator $\mathrm{P}_i = |i\rangle\langle i|$ is p_i.

Among all the mixed states that we could consider, there is one that is special, for which the probability p_i in Eq. (B.2) is chosen to coincide that of the Boltzmann factor in Eq. (5.40). Indeed, this is the *thermal state*, which reads

$$\rho_\beta = \frac{1}{\mathcal{Z}}\sum_i e^{-\beta E_i}|i\rangle\langle i| = \frac{1}{\mathcal{Z}}e^{-\beta\hat{H}}\,. \tag{B.4}$$

B.2 The KMS condition

Let us now consider the two point function $\langle\phi(x_1)\phi(x_2)\rangle_\beta$ of an operator ϕ. In equilibrium, where there is time-translational invariance, it can only depend on $t = t_2 - t_1$. Then,

$$iG^{\mathrm{F}}_\beta(t;\vec{x}_1,\vec{x}_2) := \langle\phi(x_1)\phi(x_2)\rangle_\beta = \mathrm{Tr}[\rho_\beta T(\phi(x_1)\phi(x_2))]\,, \tag{B.5}$$

with $T(\cdot)$ the time-ordered product. An observable $\mathcal{O}$ in the Heisenberg picture evolves as

$$\mathcal{O}(t_0+t) = e^{iHt}\mathcal{O}(t_0)e^{-iHt} \tag{B.6}$$

We have seen already that, at finite temperature, imaginary times will appear. Thus, we consider $t = -i\tau$. Now, we want to claim that G^{F}_β is periodic in imaginary time. Taking $\tau = \tau_2 - \tau_1 \leq \beta$, the time ordering leads to

$$T(\phi\left(i(\tau_1+\beta),\vec{x}_1\right)\phi(i\tau_2,\vec{x}_2)) = \phi\left(i(\tau_1+\beta),\vec{x}_1\right)\phi(i\tau_2,\vec{x}_2) \tag{B.7}$$

and therefore

$$\begin{aligned} iG^{\mathrm{F}}_\beta(i(\tau+\beta);\vec{x}_1,\vec{x}_2) &= \frac{1}{\mathcal{Z}}\mathrm{Tr}\left[e^{-\beta H}\phi\left(i(\tau_1+\beta),\vec{x}_1\right)\phi(i\tau_2,\vec{x}_2)\right] \\ &= \frac{1}{\mathcal{Z}}\mathrm{Tr}\left[\phi\left(i(\tau_1+\beta),\vec{x}_1\right)e^{-\beta H}\phi(i\tau_2,\vec{x}_2)\right] \\ &= \frac{1}{\mathcal{Z}}\mathrm{Tr}\left[\phi\left(i\tau_1,\vec{x}_1\right)e^{-\beta H}\phi(i\tau_2,\vec{x}_2)\right] \\ &= \frac{1}{\mathcal{Z}}\mathrm{Tr}\left[e^{-\beta H}\phi(i\tau_2,\vec{x}_2)\phi\left(i\tau_1,\vec{x}_1\right)\right] \\ &= \mathrm{Tr}[\rho_\beta T(\phi(x_1)\phi(x_2))] = iG^{\mathrm{F}}_\beta(i\tau;\vec{x}_1,\vec{x}_2)\,. \end{aligned} \tag{B.8}$$

This is the so calles Kubo–Martin–Schwinger (KMS) condition: correlation functions at finite temperature $T = 1/\beta$ are periodic in imaginary time with periodicity β.

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