Exam: The Witten soliton

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Instructions for the exam: *You are allowed to use any material you find useful (papers, books, AI...). You may also discuss it in teams. However you do it, you have to deliver your own hand-written solutions either on Campuswire or sending them to the email from above. To prepare your delivery, you can either scan your pen- or pencil-written solutions or use a digital device on which you can write by hand. Only .pdf files are allowed.*

This exam is based on Ref. [1]. Consider the following three different solutions to Einstein's equations with a negative cosmological constant $\Lambda = -6/L^2$ in five dimensions, whose line element can be written as

$$\mathrm{d}s_5^2 = \frac{L^2}{z^2}\left[-f(z)\,\mathrm{d}t^2 + \frac{\mathrm{d}z^2}{g(z)f(z)} + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + g(z)\mathrm{d}y^2\right] . \tag{1.1}$$

In all solutions, t, x^1, $x^2 \in \mathbb{R}$ are extended coordinates, while $y \in (0, \ell_y)$ is a compact periodic direction. The three different solutions are:

- An *empty* solution, in which $f(z) = g(z) = 1$ and $z \in (0, \infty)$,
- A *black brane* solution, in which $f(z) = 1 - z^4/z_\mathrm{H}^4$ and $g(z) = 1$ and $z \in (0, z_\mathrm{H})$ and
- A *soliton* solution, in which $f(z) = 1$ and $g(z) = 1 - z^4/z_0^4$ and $z \in (0, z_0)$.

Exercise 1. *Geometry.* Let us understand the different geometries.

1. Explain why for the black brane solution $z = z_\mathrm{H}$ is a horizon. In particular, compute the temperature as a function of z_H by continuing to Euclidean time $t \mapsto -i\tau$ with $\tau \in (0, \beta)$ and imposing regularity (absence of a conical singularity) at $z = z_\mathrm{H}$.

2. Explain why for the soliton and the empty solution there is no horizon. Show that, nonetheless, regularity at $z = z_0$ relates ℓ_y and z_0 in the soliton solution.

3. What is the topology of the conformal boundary for these three solutions in Lorentzian signature? And in Euclidean signature? From this, argue that when they are considered with the same value of β and the same value of ℓ_y, they describe different states of the same theory at the same temperature. What theory?

Exercise 2. *Thermodynamics.* Is there any phase transition between these solutions? To answer this question,

1. Compute the free energy of the black brane solution and the soliton solution by evaluating the Euclidean on-shell action. Use the background subtraction method, using the free energy of the empty solution as the regulator. Recall that you will have to make sure that the size of the physical circles (given by τ and y) coincide at the cut-off, before pusing the cut-off to the boundary.

2. Observe that there is a phase transition in the system. Give the value of the critical temperature T_c in units of ℓ_y. Which phase is dominant at every temperature? Is any of the three solutions never dominant?

3. Plot the free energy density as a function of the temperature in units of ℓ_y.

4. Compute the entropy of the different solutions and plot it as a function of the temperature in units of ℓ_y. State how the entropy scales with N in the different phases and explain in what sense *i)* there is a confinement/deconfinement phase transition and *ii)* the size of the circle ℓ_y sets the confinement scale.

Exercise 3. *Wilson loop.* In this exercise you are asked to compute the quark-antiquark potential of a pair of quarks separated a distance d in the x^1 direction, and that sit at the same point in x_2 and y. Use as a regulator the length of a straight string stretching from $z = 0$ to $z = \infty$ in the empty geometry.

1. Compute the potential between quarks in the soliton solution. Plot it in units of the critical temperature T_c. What is happening to the string at large separations?

2. Compute the potential between quarks in the black brane geometry. Note that, since the coordinate y acts as an spectator, this is quite close to what we discussed in Lecture 4 (the difference is that now we are using a slightly different regulator).

3. Plot the quark-antiquark potential you found in item (2.) at three different temperatures —for example, $T = T_c$, $2T_c$ and $10T_c$— on top of the plot you produced in (1.). Show that, in units of T_c, the separation at which the string breaks increases with the temperature. Give a physical explanation for this phenomenon.

4. Compare the qualitative features of the plot you just drew to Ref. [2]. Say in what sense the model shares some properties with QCD. However, note that the mass of the particles in the spectrum scales as $M_n \simeq \ell_y^{-1}$ (you do not have to show this, see Ref [3]). In particular it does not scale with N and λ. Compare it to the slope of the potential and comment why this is unsatisfactory.

Useful expressions

$$\int_0^{\zeta_*} \frac{\zeta^2\, \mathrm{d}\zeta}{\sqrt{(\zeta^4-1)(\zeta^4-\zeta_*^4)}} = \zeta_* \frac{\sqrt{\pi}\Gamma(7/4)}{3\Gamma(5/4)}\, {}_2F_1\left(\frac{1}{2},\frac{3}{4};\frac{5}{4};\zeta_*^4\right),$$
$$\int_0^{\zeta_*}\left[\frac{\zeta_*^2}{\sqrt{(\zeta^4-1)(\zeta^4-\zeta_*^4)}}-1\right]\frac{\mathrm{d}\zeta}{\zeta^2} = \frac{1}{\zeta_*} + \frac{\sqrt{\pi}\Gamma(7/4)\left(\left(\zeta_*^4-1\right)\,{}_2F_1\left(\frac{1}{2},\frac{3}{4};\frac{1}{4};\zeta_*^4\right) - {}_2F_1\left(-\frac{1}{2},\frac{3}{4};\frac{1}{4};\zeta_*^4\right)\right)}{6\zeta_*\Gamma(5/4)}. \tag{1.2}$$

References

[1] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[2] O. Kaczmarek, F. Karsch, P. Petreczky and F. Zantow, *Heavy quark anti-quark free energy and the renormalized Polyakov loop*, *Phys. Lett. B* **543** (2002) 41 [hep-lat/0207002].

[3] N. R. Constable and R. C. Myers, *Spin two glueballs, positive energy theorems and the AdS / CFT correspondence*, *JHEP* **10** (1999) 037 [hep-th/9908175].