Week 1: Anti de Sitter space

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The objective of this exercise sheet is that you familiarize with the properties of anti de Sitter (AdS) space and understand the claims that are often made about it.

Exercise 1. *Construction of AdS.* Consider the $D+1$ dimensional space $\mathbb{R}^{2,D-1}$, with coordinates $X^0, \cdots, X^D$ and endowed with the metric,

$$\mathrm{d}s^2_{D+1} = \eta_{AB}\mathrm{d}X^A\mathrm{d}X^B = -(\mathrm{d}X^0)^2 + (\mathrm{d}X^2)^2 \cdots - (\mathrm{d}X^D)^2 \,, \tag{1.1}$$

(note the two time directions, X^0 and X^D). Plot the D-dimensional hypersurface determined by

$$-(X^0)^2 + (X^1)^2 + \cdots + (X^{D-1})^2 - (X^D)^2 = -L^2 \,, \tag{1.2}$$

using three axes. To do so, identify the usual "x-axis" with X^0, the "z-axis" with X^D and gather the rest of the dimensions in the "y-axis".

Exercise 2. *Global coordinates in AdS.* Parametrize the space you have just constructed using the coordinate system

$$X^0 = L\cosh\frac{\rho}{L}\sin\frac{\tau}{L}\,, \qquad X^k = L\,n^k\sinh\frac{\rho}{L}\,, \qquad X^D = L\cosh\frac{\rho}{L}\cos\frac{\tau}{L}\,, \tag{1.3}$$

with n^k parametrizing a $(D-2)$-dimensional unit sphere, $\sum(n^k)^2 = 1$. These are the so-called *global coordinates*.

1. Check that this choice does indeed satisfy Eq. (1.2), and write down the embedded metric in these coordinates.

2. What are the constant-τ slices? And the constant-ρ slices? Which one is a time direction? Specify their topology and draw them on the hypersurface.

3. Perform one more change of coordinate $r = L\sinh(\rho/L)$. If you did everything right you should have obtained the metric

$$\mathrm{d}s^2_D = -\left(\frac{r^2}{L^2}+1\right)\mathrm{d}\tau^2 + \left(\frac{r^2}{L^2}+1\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{(D-2)}\,, \tag{1.4}$$

with $\mathrm{d}\Omega^2_{(D-2)}$ the line element on the $(D-2)$ unit sphere. Note that in Eq. (1.3) we have $\tau \in (0, 2\pi)$ and $\rho \in (-\infty, +\infty)$. Often we will consider the universal covering space instead, and take $\tau \in (-\infty, \infty)$.

Exercise 3. *Symmetries.* From now on we set $D = 5$. Convince yourself that AdS_5 is a maximally symmetric space with isometry group SO(4, 2). You can use the isometries generated by the rotational ($\equiv$ preserving η_{AB}) Killing vectors of the ambient space $\mathbb{R}^{2,4}$,

$$J_{AB} = \eta_{AC}X^C\partial_B - \eta_{BC}X^C\partial_A = -J_{BA}\,, \tag{1.5}$$

which provide a realization of the Lie algebra of the isometry group. In particular, note that $\partial\tau$ is a timelike Killing vector. To which of the J_{AB}'s does it correspond?

Exercise 4. *Conformal coordinates, conformal boundary and Penrose diagram.* Consider the change of coordinates given by

$$\cosh \frac{\rho}{L} = \frac{1}{\cos\theta} \,. \tag{1.6}$$

1. Find the range of θ (recall $\rho \in (-\infty, +\infty)$. Which value of $\theta = \theta_b$ corresponds to the *boundary* ($\rho \to \infty$) of AdS?

2. After performing the change of variables (1.6), show that the metric takes the form

$$g_{\mu\nu} = \Omega(\theta)^2 \, \tilde{g}_{\mu\nu} \,, \tag{1.7}$$

   with $\Omega(\theta_b) \to \infty$ while $\tilde{g}_{\mu\nu}$ stays finite. What is the topology of the boundary?

3. Draw the Penrose diagram of AdS. You may find it helpful to consider light-rays moving on the (τ, θ)-plane.

Exercise 5. *AdS is a box.* Consider a light ray thrown from the center of AdS ($\rho = 0$) towards the boundary, moving along the ρ direction (fixed angles).

1. Show that such a ray reaches the boundary in a finite time, after $\Delta\tau = \pi L/2$.

2. Repeat the exercise for a massive particle, and observe that geodesics are given by the intersection of AdS with 2-planes through the origin of the embedding space. The easiest way to do this is:

   - observe that a particle sitting at rest at $\rho = 0$ follows a geodesic,
   - perform a boost in the directions X^0, X^5; keeping X^i fixed for $i = 1, \cdots, 4$; and
   - conclude that the geodesic is given by

$$\tanh\rho = \tanh\beta \, \cos\tau \,. \tag{1.8}$$

3. Explain in what sense AdS acts like a box.

Exercise 6. *Poincaré coordinates and the Poincaré patch.* Let us now consider a different set of coordinates defined by

$$X^\alpha = r x^\alpha \,, \qquad X^5 - X^4 = r \,, \qquad X^5 + X^4 = r^{-1} + r\eta_{\alpha\beta} x^\alpha x^\beta \,. \tag{1.9}$$

Here, the Greek indexed α, $\beta = 0, \cdots, 4$; $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ is the four dimensional Minkowski metric and $r > 0$ is a radial coordinate.

1. Check that these coordinates satisfy Eq. (1.2) and find the metric in this basis.

2. Show that these coordinates make scaling symmetry explicit, $(x^\alpha, r) \mapsto (\mu x^\alpha, \mu^{-1} r)$.

3. Introducing the variable $z = r^{-1}$, show that the metric is conformally flat. What is the topology of the conformal boundary now?

4. What part of the Penrose diagram is covered by the Poincaré patch?

Exercise 7. *AdS has constant negative curvature.* Convince yourself that AdS is a space with constant negative curvature, and consequently solves Einstein's equations with a negative cosmological constant Λ. Give the value of Λ in terms of the AdS radius L.