Week 2: The Hawking–Page phase transition

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The objective of this exercise sheet is to help you read and understand Ref. [1], and its importance in holography. See also Ref. [2].

Exercise 1. *The Schwarzschild-AdS black hole.* Consider the following geometry,

$$\mathrm{d}s_5^2 = -f(r)\mathrm{d}\tau^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2\,, \qquad f(r) = \frac{r^2}{L^2} + 1 - \frac{\mu L^2}{r^2}. \tag{1.1}$$

This is a black hole geometry with constant negative Ricci scalar $R = -20/L^2$. Find the position of the horizon, r_{H}, in terms of the parameter μ. Show that it satisfies

$$\mu = \frac{r_{\mathrm{H}}^2}{L^2} + \frac{r_{\mathrm{H}}^4}{L^4}\,. \tag{1.2}$$

Because we are interested in the thermodynamic properties of this solution, we work in Euclidean signature, $\tau \mapsto -i\tau_E$. In the semiclassical approximation to Euclidean quantum gravity, the thermodynamical partition function $\mathcal{Z}$ and the free energy F are given by

$$\mathcal{Z}[\beta] = e^{-\beta F} \simeq e^{-I_E[g^{(cl)}]}\,, \tag{1.3}$$

where $I_E[g^{(cl)}]$ is the Euclidean action of a classical solution $g^{(cl)}$ with periodic boundary conditions in imaginary time $\tau_E \in (0,\beta)$, with $\beta = 1/T$. Often, this action is divergent and needs renormalization. One way to do it is by subtracting the action $I_E[g^{(0)}]$ of a reference spacetime $g^{(0)}$ with the same asymptotics.

Exercise 2. *Conformal boundary of the Schwarzschild-AdS black hole.* Show that Eq. (1.1) has the same asymptotics as global AdS. What is, then, the conformal boundary?

Exercise 3. *Computation of the temperature.* For that,

1. Expand the metric (1.1) near the horizon, $r \simeq r_{\mathrm{H}}$, and notice that in the coordinate ρ defined by
$$\rho^2 = \frac{2L^2 r_{\mathrm{H}}}{L^2 + 2r_{\mathrm{H}}^2}(r - r_{\mathrm{H}})\,, \tag{1.4}$$
the $(\tau_{\mathrm{E}}, \rho)$ part of the metric resembles $\mathbb{R}^2$ written in polar coordinates.

2. Fix the value of β to avoid a conical singularity at $\rho = 0$. This fixes the temperature of the black hole.

3. What is the behavior of the temperature as a function of the position of the horizon? Explain whether there is any extremum, and what happens in the $r_{\mathrm{H}} \to 0$ and $r_{\mathrm{H}} \to \infty$ limits.

Exercise 4. *Computation of the entropy.* Calculate the entropy of the black hole. Recall that it is given by the area of the horizon via

$$S_{\mathrm{BH}} = \frac{A}{4G_5} = \frac{2\pi A}{\kappa_5^2}\,. \tag{1.5}$$

Exercise 5. *Computation of the free energy.* We will now compute the free energy by evaluating Eq. (1.3), where the Euclidean action is

$$I_E = -\frac{1}{2\kappa_5^2}\int_M \mathrm{d}^5x\sqrt{g}\Big(R+\frac{12}{L^2}\Big) - \frac{1}{\kappa_5^2}\int_{\partial M}\mathrm{d}^4x\sqrt{h}K\,. \tag{1.6}$$

1. Observe that the on-shell Euclidean action is divergent. Regularize it by subtracting the action $I_E[g^{(0)}]$ of global AdS, which we denote by $g^{(0)}$. To do so,

   - take the boundary hypersurface ∂M to be defined by the constant slice at $r = r_c$, which we will eventually send to infinity and
   - make sure that both metrics on the hypersurface match. In particular, impose that the length of the temporal circle is the same,

   $$\int_0^\beta \mathrm{d}\tau\sqrt{h_{\tau\tau}} = \int_0^{\beta_0}\mathrm{d}\tau\sqrt{h^{(0)}_{\tau\tau}}\,. \tag{1.7}$$

2. Find the entropy and the energy from the usual thermodynamic relations

   $$S = -\frac{\mathrm{d}F}{\mathrm{d}T}\,,\qquad E = \frac{\mathrm{d}(\beta F)}{\mathrm{d}\beta} = F + TS\,. \tag{1.8}$$

   Observe that the entropy coincides with what you got from Eq. (1.5).

Exercise 6. *Phase diagram of Schwarzschild-AdS.* Draw separately the energy, the entropy and the free energy as a function of the temperature. Then

1. Discuss the stability of the different branches of solutions. Is there any phase transition in the system?
2. Do these black holes evaporate? What does it depend on?
3. What happens in the $r_{\mathrm{H}} \gg L$ limit? And in the $r_{\mathrm{H}} \ll L$ limit?

Exercise 7. *Holographic interpretation.* If we trust holography, this solution is describing the physics of strongly coupled $\mathcal{N} = 4$ on the boundary.

1. Use the "holographic dictionary" to change from the gravity variables κ_5^2 and L to the gauge theory quantities N and λ.
2. How does the entropy scale with N at high temperatures? And at low temperatures?
3. In what sense is the system describing a confinement/deconfinement phase transition? Why is there a "confined" phase?

References

[1] S. W. Hawking and D. N. Page, *Thermodynamics of Black Holes in anti-De Sitter Space*, *Commun. Math. Phys.* **87** (1983) 577.

[2] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].