Week 3: fields in the bulk and operators at the boundary

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The objective of this exercise sheet is to understand how fields in the gravity side are related to operators of the gauge theory.

Exercise 1. *A massive scalar in AdS.* For simplicity, we work in Euclidean signature. Consider a massive scalar field Φ with action

$$S = \frac{1}{2\kappa_{d+1}^2}\frac{1}{2}\int \mathrm{d}^{d+1}x\,\sqrt{g}\left(g^{MN}\partial_M\Phi\partial_N\Phi + m^2\Phi^2\right), \tag{1.1}$$

in a fixed pure AdS background (Poincaré patch),

$$ds^2 = \frac{L^2}{z^2}\left(\mathrm{d}z^2 + \delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right), \qquad \mu = 1,\ldots,d. \tag{1.2}$$

Note that we are using the coordinate $z = 1/r$, in terms of which the boundary of AdS is at $z = 0$.

1. Write down the equation of motion of the field.
2. Perform a mode expansion

$$\Phi(z,x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{ik\cdot x}\,\Phi_k(z), \qquad k \equiv |k|, \tag{1.3}$$

   and obtain the ordinary differential equation satisfied by each mode $\Phi_k(z)$.
3. Assume that $\Phi_k(z) \sim z^\alpha$ close to the boundary. What are the possible values of α? Explain how these are related to the dimension Δ of the operator $\mathcal{O}$ dual to Φ. Is there any special value of m^2?
4. Specify the result for the case in which Φ is the dilaton in AdS_5 (so $d = 4$). What is the dimension of the dual operator?
5. Use your knowledge on Bessel functions (or software) to show that the general solution to the mode equation is

$$\Phi_k(z) = z^{d/2}\left(A(k)\,I_\nu(kz) + B(k)\,K_\nu(kz)\right). \tag{1.4}$$

   What is ν in terms of the mass m and the boundary dimension d?
6. Observe that regularity in the interior ($z \to \infty$) fixes one of the integration constants. Which one, and why?
7. In the next exercise we will encounter divergences close to the boundary, and for this reason we introduce a cutoff at $z = \varepsilon$. Impose the boundary condition at $z = \varepsilon$,

$$\Phi_k(\varepsilon) = \varepsilon^{\Delta_-}\,\phi_{(0)}(k), \qquad \Delta_- = \frac{d}{2} - \nu, \tag{1.5}$$

   find $B(k)$ and give an explicit expression for $\Phi_k(z)$. Note that, defined in this way, $\phi_{(0)}(k)$ coincides with the Fourier transform of the coefficient accompanying the leading near-boundary behaviour $\Phi(z,x) \sim z^{\Delta_-}\phi_{(0)}(x)$.

Exercise 2. *One- and two-point functions.* From now on, assume that ν is not an integer.

1. Integrating by parts and using the equations of motion, show that the on-shell action $S_{\text{o-s}}$ reduces to a boundary term at $z = \varepsilon$.

2. Show that, substituting our solution, the on-shell action takes the form

$$S_{\text{o-s}}(\varepsilon) = \frac{L^{d-1}}{2\kappa_{d+1}^2} \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \varepsilon^{-2\nu+1} \, \phi_{(0)}(k) \, \phi_{(0)}(-k) \, \left[\partial_z \log\Big(z^{d/2} K_\nu(kz) \Big) \right]_{z=\varepsilon} . \quad (1.6)$$

3. From the small-z expansion of the Bessel function, one can show that

$$\partial_z \log\Big(z^{d/2} K_\nu(kz) \Big) = \frac{\Delta_-}{z} + \frac{k^2 z}{2(1-\nu)} - \frac{k^4 z^3}{8(1-\nu)^2(2-\nu)} + \cdots + 2\nu \, \frac{b}{a} \, k^{2\nu} \, z^{2\nu-1} + \cdots . \quad (1.7)$$

   Observe that the on-shell action is divergent as $\varepsilon \to 0$, and can be renormalized by adding local counterterms on the cutoff surface:

$$S_{\text{ct}}(\varepsilon) = \frac{1}{2\kappa_{d+1}^2} \int_{z=\varepsilon} \mathrm{d}^d x \, \sqrt{\gamma} \, \Big[c_1 \, \Phi^2 + c_2 \, (\nabla_\gamma \Phi)^2 + c_3 \, (\nabla_\gamma^2 \Phi)^2 + \cdots \Big] , \quad (1.8)$$

   where $\gamma^{\mu\nu}$ is the induced metric on the $z = \varepsilon$ hypersurface and ∇_γ its covariant derivative. How many counterterms are needed as a function of ν? Find the values of the first three c_i's.

4. Note that the last (non-local) term in Eq. (1.7) gives a finite contribution to the on-shell action. Define the renormalized on-shell action as

$$S_{\text{ren}} = \lim_{\varepsilon \to 0} \Big(S_{\text{o-s}}(\varepsilon) + S_{\text{ct}}(\varepsilon) \Big). \quad (1.9)$$

5. Correlation functions are obtained by functional differentiation of $S_{\text{ren}}[\phi_{(0)}]$ with respect to the source. Compute the one-point function $\langle \mathcal{O}(k) \rangle_{\phi_{(0)}}$ and the two-point function $\langle \mathcal{O}(k) \mathcal{O}(-k) \rangle$ (up to contact terms).

6. Show that the momentum-space correlator scales as $k^{2\nu}$ and hence in position space behaves as $|x|^{-2\Delta}$.