Week 4: shear viscosity to entropy density ratio

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In this exercise you will find that the shear viscosity to entropy density ratio is a universal constant for any five dimensional black brane geometry, which we write in the form

$$\mathrm{d}s_5^2 = e^{2A}\left[f(u)^{-1}\mathrm{d}u^2 - f(u)\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right]. \tag{1.1}$$

We assume that the manifold is asymptotically AdS, and that its boundary is at $u = 0$. Similarly, there is a horizon at $u = u_{\mathrm{H}}$, where $f(u_{\mathrm{H}}) = 0$.

Exercise 1. *Equation of motion.* To compute the shear viscosity we are interested in considering metric perturbations $g_{\mu\nu} \mapsto g_{\mu\nu} + h_{\mu\nu}$ and, more precisely, h_{xy}. It turns out that its equation decouples from the rest, and becomes that of a massless scalar,

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\big(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\,h_{xy}\big) = 0\,. \tag{1.2}$$

Show that the Fourier modes with momentum k along the z direction,

$$h_{xy} = \int \frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^2}\,\phi_{\omega,k}(u)\,e^{-i\omega t + ikz}\,, \tag{1.3}$$

satisfy the equation

$$\phi''_{\omega,k} + \left(3A' + \frac{f'}{f}\right)\phi'_{\omega,k} + \frac{\omega^2 - fk^2}{f^2}\,\phi_{\omega,k} = 0\,. \tag{1.4}$$

To simplify notation, we will drop the subscripts ω, k in what follows, writing $\phi_{\omega,k} = \phi$.

Exercise 2. *Asymptotic analysis.* Show that close to the AdS boundary (i.e. for small values of the radial coordinate u) Eq. (1.4) admits the following asymptotic solution

$$\phi \simeq a(\omega,k) + b(\omega,k)\,u^4\,. \tag{1.5}$$

For that, use that $A \simeq \log(L/u)$ and $f \simeq 1$ close to the boundary, and assume that $\phi \simeq u^\alpha$. Then write down the leading order behavior of Eq. (1.4) and find the allowed values of α.

Exercise 3. *Low energy analysis.* Show that, for $k = 0$ and in the low frequency limit $\omega \to 0$, the solution to Eq. (1.4) can be written as

$$\phi(u) = c_1 + c_2\int_0^u \frac{e^{-3A}}{f}\,\mathrm{d}\overline{u} \tag{1.6}$$

1. Argue that the constant c_2 in Eq. (1.6) must vanish in the exact $\omega = 0$ limit. In particular, it has to be proportional to the frequency ω.

2. Show that $a(\omega,0) = c_1$ and $b(\omega,0) = -c_2/4$ by expanding Eq. (1.6) close to the boundary.

Exercise 4. *Near horizon analysis.* We study now the solution close to the horizon, located at $u = u_{\mathrm{H}}$ such that $f = f'(u_{\mathrm{H}})(u - u_{\mathrm{H}}) + O\big((u-u_{\mathrm{H}})^2\big)$ and $A = A_{\mathrm{H}} + O(u - u_{\mathrm{H}})$.

1. Show that near the horizon, Eq. (1.4) takes the leading form

$$\phi'' + \frac{1}{(u-u_{\rm H})}\,\phi' + \frac{\omega^2}{\big(f'(u_{\rm H})\big)^2 (u-u_{\rm H})^2}\,\phi = 0 \qquad (1.7)$$

2. Assuming the form $\phi = (u-u_{\rm H})^\gamma(1+\ldots)$ close to the horizon, show that Eq. (1.7) admits two solutions

$$\phi(u) = (u-u_{\rm H})^{\pm i\omega/f'(u_{\rm H})}\, F(u,\omega) \qquad (1.8)$$

with $F(u,\omega)$ some regular function at the horizon.

Exercise 5. *Ingoing boundary conditions.* We need to choose the sign in Eq. (1.8) that corresponds to waves falling into the horizon. To understand which is the correct sign,

1. Perform the change of coordinates

$$\mathrm{d}u^* = f^{-1}\mathrm{d}u\,, \qquad \mathrm{d}v_\pm = \mathrm{d}t \pm \mathrm{d}u^*\,, \qquad (1.9)$$

to write the metric as

$$\mathrm{d}s^2 = e^{2A}\left(-f\,\mathrm{d}v_\pm^2 \pm 2\,\mathrm{d}u\,\mathrm{d}v_\pm + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)\,. \qquad (1.10)$$

This change leaves the metric in the so-called Eddington–Finkelstein coordinates, in which the metric has a regular behavior at the horizon. The $v_\pm$ are the two Eddington–Finkelstein null times (one ingoing, one outgoing). In the next steps you will determine which is which by analyzing the near-horizon wave.

2. Restore the time dependence in the perturbation and use Eq. (1.8) together with the tortoise coordinate u^* in Eq. (1.9) to show that the time dependent perturbation takes the form

$$h_{xy} = e^{-i\omega(t\mp u^*)}\,F(u^*,\omega) \qquad (1.11)$$

3. Determine which of these two solutions correspond to infalling boundary conditions.

Exercise 6. *Matching procedure.* Eqs. (1.6) and (1.8) have an overlapping region. Identify the near horizon expansion of the former with the low energy limit of the latter to find c_1 and c_2 in terms of the horizon value at zero frequency of the function F defined by Eq. (1.8), that is $F(u_{\rm H},0)$.

Exercise 7. *Shear viscosity to entropy density ratio.* The retarded Green's function appearing in the Kubo formula is given in terms of $a(\omega,k)$ and $b(\omega,k)$, thanks to the holographic dictionary, as

$$G^R(\omega,k) = \frac{1}{4\pi G_5}\,\frac{b(\omega,k)}{a(\omega,k)}\,, \qquad (1.12)$$

in our conventions, and with G_5 the five-dimensional Newton's constant.

1. Find the value of the shear viscosity η in terms of the coefficient $A_{\rm H}$.

2. Using Bekenstein–Hawking entropy formula, give the ratio between shear viscosity η and entropy density s.

Based on an exercise prepared by Prof. Umut Gursoy for the PhD Summer School in 2019.