2 Holography made concrete

In the previous section we saw some pieces of evidence that gravity has a holographic nature, and that amplitudes of strings and gauge theories share similarities, hinting towards some sort of gauge/string duality. In this section we will see how holography is realized in the AdS/CFT correspondence. But before that, we need to uncover a key object that was left aside in the previous section and plays a crucial role in the realization of the duality —Dirichlet branes.

2.1 Open strings and D-branes

In the previous section we discussed that a particularly convenient way to write down the string action is the Polyakov action. We also observed that the massless excitations of the closed string realize three fields on the target space: a symmetric tensor which we naturally identify with the metric, $G_{\mu\nu}$, a 2-form $B_{\mu\nu}$ and the dilaton Φ, a scalar field whose asymptotic value is related to the string coupling constant g_s.

We should now discuss open strings. These will consist of a worldsheet with a boundary, and thus we have to be careful about the variation of the fields at the boundary. More precisely, the variation of the Polyakov action contains a boundary term

$$\delta S_{\text{bdry}} = -\frac{1}{2\pi\ell_s^2}\int_{\partial\Sigma} \mathrm{d}\tau\, \delta X^\mu \Big(\eta_{\mu\nu}\,\partial_\sigma X^\nu\Big)\,. \tag{2.1}$$

Thus, for open strings we need extra conditions on the endpoints of the string. The boundary conditions at the endpoints have to be either

- **Neumann:** the endpoint is free to move in the direction "μ",

$$\partial_\sigma X^\mu = 0\,. \tag{2.2}$$

- **Dirichlet:** the endpoint is fixed,

$$\delta X^\mu = 0\,. \tag{2.3}$$

Now, we can choose that the coordinates $X^0, \cdots, X^p$ satisfy Neumann boundary conditions, while $X^{p+1}, \cdots, X^{D-1}$ satisfy Dirichlet. Then the corresponding endpoint is constrained to move on a $(p+1)$-dimensional hypersurface in spacetime, which we call a *Dp-brane*. Thus, at this point, a D-brane is just a defect in spacetime on which the endpoints of open strings can be attached and move.

But string theory "gives life" to these objects. To see this, let us discuss the spectrum of open strings. They have their own massless sector, analogous to the massless fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ that emerge from closed strings. The massless spectrum in this case contains

- A spin-1 field *on the brane* that we identify with the gauge field A_a, with $a = 1, \cdots, p$ lying on the brane.
- A set of $D-p$ scalar fields ϕ^I living on the brane, interpreted as fluctuations of the brane in the transverse directions $I = p+1, \cdots, D-1$.

As we mentioned, the fact that the strings are open introduces boundaries in the worldsheet of the string. But on this worldsheet we still have a two-dimensional CFT. The scattering of open-string states, for example, is now mapped to a disk with vertex operators inserted on the boundary, as in Fig. 1.

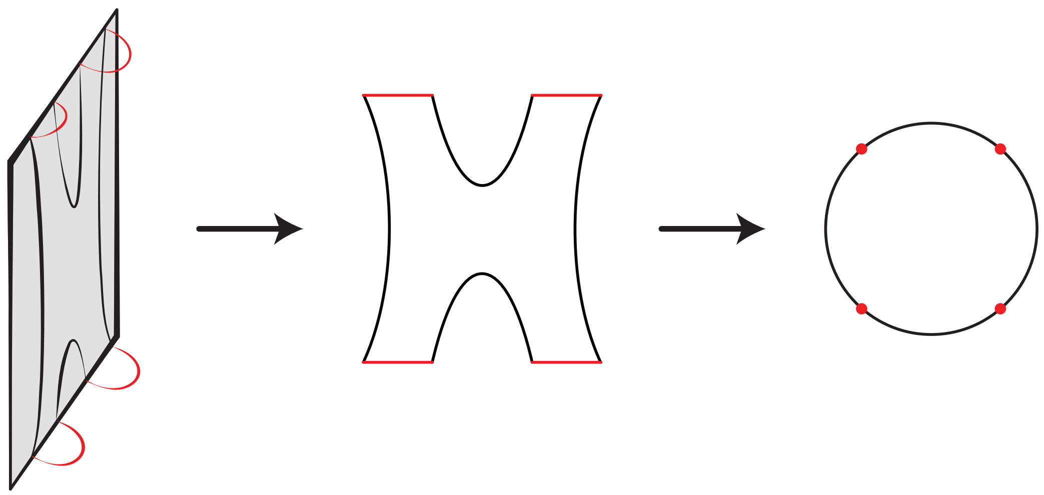

Figure 1: Four-point scattering of open strings. The process is conformal to a disk with four insertion operators. The boundary of the disk corresponds to the endpoints of the propagating strings, attached to the brane.

Remarkably, the presence of these scalar fields, identified as transverse fluctuations, already hints at the fact that D-branes should indeed be treated as dynamical objects.

More importantly, as soon as the brane is included as a hypersurface where open strings can be attached, it necessarily gravitates. Indeed, imagine we have two D-branes between which open strings can be attached. It turns out that this forces us to admit that the branes can interchange gravitons, since such an interaction is equivalent to an open-string loop —see Fig. 2.

The correct way to describe the dynamics of the brane is the Dirac–Born–Infeld action,

$$S_{\text{DBI}} = -T_p \int \mathrm{d}^{p+1}\xi \, \sqrt{-\det\left(\gamma_{ab} + 2\pi\alpha' F_{ab}\right)}\,, \tag{2.4}$$

with ξ coordinates along the brane, γ_{ab} the induced metric on the brane

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu}\,, \tag{2.5}$$

and F_{ab} the field strength associated to the massless gauge fields that appear in the spectrum of the open string. This is a fairly complicated action —it is highly non-linear—. Nonetheless, if we work at low energies, we can choose $X^a = \xi^a$, in which case we have

$$\gamma_{ab} = \eta_{ab} + \frac{\partial X^I}{\partial \xi^a}\frac{\partial X^J}{\partial \xi^b}\delta_{IJ}\,, \tag{2.6}$$

then the action can be expanded in powers of ℓ_s. Writing $\phi^I = X^I/(2\pi\ell_s^2)$ we obtain

$$S = -(2\pi\ell_s^2)^2 T_p \int \mathrm{d}^{p+1}\xi \left(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial_a\phi^I\partial^a\phi^I + \ldots\right)\,, \tag{2.7}$$

which is simply free Maxwell theory coupled to free massless scalar fields.

A further important point is that a Dp-brane is not an arbitrary defect: its tension T_p is *fixed* by string theory. A clean way to see this is to compare the interaction between two parallel Dp-branes computed in the two equivalent channels depicted in Fig. 2. In the open-string channel it is given by the one-loop vacuum amplitude of an open string stretched between the branes (the

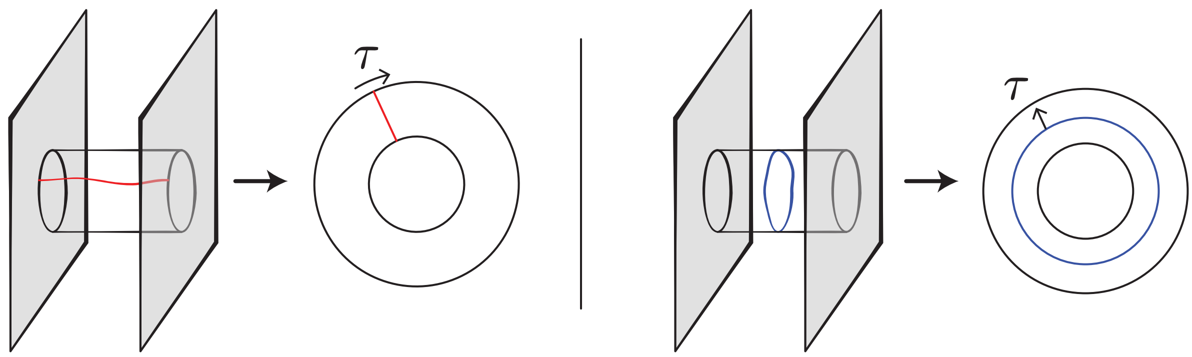

Figure 2: (Left) The open-string vacuum loop diagram is conformal to the annulus. (Right) The same diagram can be interpreted as a closed-string exchange between two D-branes.

annulus diagram). By worldsheet channel duality, the same annulus can be reinterpreted as a tree-level exchange of closed-string modes propagating between the two branes. Matching the long-distance force extracted from the closed-string channel with the normalization of the open-string amplitude fixes the overall coupling of the brane to bulk fields and therefore determines the tension,

$$T_p = \frac{1}{\ell_s g_s (2\pi\ell_s)^p} \,. \tag{2.8}$$

In addition, in Type II superstring theory D-branes are BPS objects that carry Ramond–Ramond charge: they act as sources for a $(p+1)$-form RR potential C_{p+1}. Correspondingly, the low-energy brane action contains, in addition to the DBI term, a Chern–Simons coupling of the schematic form

$$S_{\rm CS} \sim \mu_p \int_{\rm Dp} C \wedge e^{2\pi\ell_s^2 F + B} \,, \tag{2.9}$$

with μ_p the RR charge (equal to the tension for BPS branes in appropriate units). Physically, a single Dp-brane carries *one unit* of RR charge, i.e. it sources one unit of the corresponding RR flux; therefore a stack of N coincident Dp-branes carries N units of RR charge and sources N units of flux. This RR charge is precisely what allows stacks of D3-branes to source the self-dual five-form flux F_5 that will appear in the supergravity description below.

2.2 A stack of N coincident branes and the AdS/CFT correspondence

We are now ready to understand how the AdS/CFT correspondence was proposed. Imagine we want to describe a system with N coincident D3-branes in Type IIB superstring theory. On the one hand, we know these can be thought of as defects in ten-dimensional spacetime to which open strings can be attached. But we also learned that they are dynamical objects: consequently, they should bend spacetime according to the laws of general relativity. It turns out that there is indeed a solution to Type IIB supergravity which can be identified as the spacetime backreaction of such a collection of branes. Explicitly, the geometry is

$$\begin{aligned} {\rm d}s^2 &= \left(1+\frac{L^4}{r^4}\right)^{-\frac{1}{2}} \eta_{\alpha\beta}{\rm d}x^\alpha {\rm d}x^\beta + \left(1+\frac{L^4}{r^4}\right)^{\frac{1}{2}} \delta_{ij}{\rm d}y^i{\rm d}y^j \,, \\ &= \left(1+\frac{L^4}{r^4}\right)^{-\frac{1}{2}} \eta_{\alpha\beta}{\rm d}x^\alpha {\rm d}x^\beta + \left(1+\frac{L^4}{r^4}\right)^{\frac{1}{2}} \left({\rm d}r^2 + r^2 {\rm d}\Omega_{(5)}\right) \end{aligned} \tag{2.10}$$

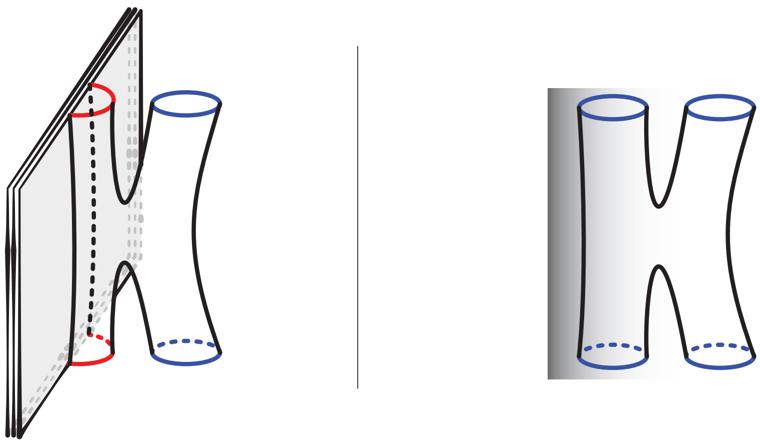

Figure 3: Two possible descriptions of a stack of N coincident D-branes: as a defect in spacetime (left) where open strings are attached, or as a solitonic geometry produced by the branes (right). The two scattering processes represented are supposed to be physically equivalent.

where $x^0, \cdots, x^3$ and $y^4, \cdot, y^9$ are, respectively, coordinates parallel and transverse to the collection of D3-branes; and $\mathrm{d}\Omega_{(5)}$ indicates the line element of a five-dimensional unit sphere. Importantly, in this solution the dilaton is constant and there is a non-vanishing 5-form RR flux,

$$e^{\Phi} = g_s = \text{constant}\,, \qquad F_5 = 4L^4\omega_5\,, \tag{2.11}$$

with ω_5 the volume form of the five-sphere. If this geometry is indeed sourced by a set of N D3-branes, the flux F_5 must be sourced by these and it should consequently be quantized, meaning that

$$\int_{S^5} F_5 = 2\kappa_{10}^2 T_{\mathrm{D3}}\, N\,. \tag{2.12}$$

Substituting the values of Newton constant and the tension of the brane, this implies that

$$L^4 = 4\pi g_s N\, \ell_s^4. \tag{2.13}$$

This solution of Type IIB supergravity has indeed the same properties —as seen from an asymptotic observer (i.e. in the $r \to \infty$ region)— as the set of branes understood as a defect. We claim therefore that both are valid descriptions of the system. Put differently, processes involving closed strings in flat space with a defect coupled to open strings attached to the defect should be in one-to-one correspondence with processes involving just closed strings propagating in the geometry Eq. (2.10), as depicted in Fig. 3.

Let us now consider the low-energy limit of both descriptions. We perform this limit in the same way that we did earlier with open and closed strings, imagining that the string scale ℓ_s is small. More precisely, we take $\ell_s \to 0$ while keeping the energy scale of the processes we consider E, the string coupling g_s and the number of branes N finite.

- **Low-energy limit of the defect picture**. In this limit we obtain two decoupled sectors. The reason is that when one looks at processes that involve both open and closed strings (like in Fig. 3), the effective coupling at some energy scale E does not only involve g_s, but also ℓ_s. In the limit we are discussing,

$$\text{effective bulk–brane coupling} \sim g_s \ell_s^4 E^4 \to 0\,, \tag{2.14}$$

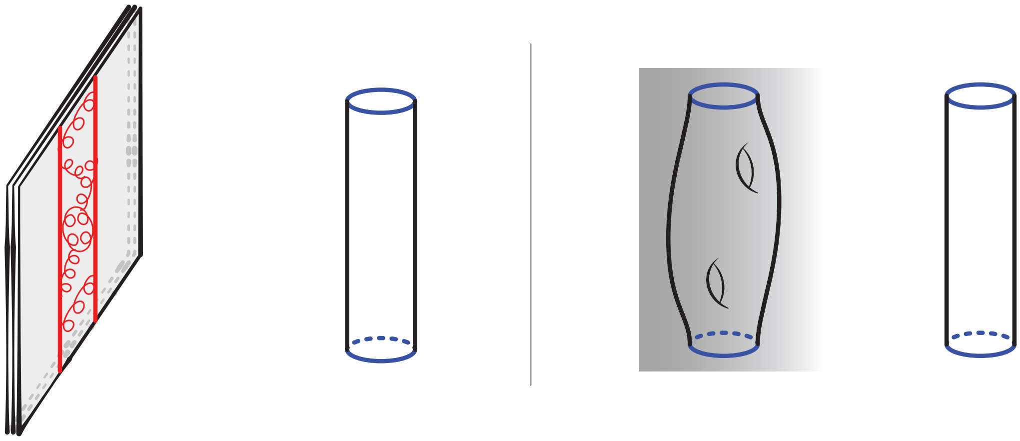

Figure 4: AdS/CFT correspondence: the decoupled low-energy open-string sector on the branes is identified with the decoupled throat sector of closed strings.

hence, they decouple. Ultimately, this is controlled by the range of the gravitational interaction in ten dimensions, $\kappa_{10}^2 \sim g_s^2 \ell_s^8$.

What are these two sectors? On the one hand, we have closed strings propagating in ten-dimensional flat space, as if the D3-branes were not present. On the other hand, we have the low-energy limit of the open strings attached to the branes. This is similar to the one we discussed previously, containing a gauge field on the brane and a set of scalar fields encoding fluctuations of the brane —see Eq. (2.7). But recall that now we consider N branes and supersymmetry. In this case the gauge group becomes non-Abelian $U(N)$, each brane representing a different color; then both the gauge fields A_μ^a and $(\phi^I)^a$ carry a color index (indicating to which brane they belong) and transform in the adjoint representation of the group. There are also Weyl fermions required by supersymmetry.

It turns out that the low-energy description of this sector becomes $\mathcal{N}=4$ supersymmetric Yang–Mills (SYM) with gauge group[1] SU(N), described by the Lagrangian

$$\mathcal{L}_{\text{SYM}} = \frac{1}{g_{\text{YM}}^2}\,\text{Tr}\left[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(D^\mu\Phi^m\right)^2 - \frac{1}{4}\left[\Phi^m,\Phi^n\right]^2 \right. \\ \left. + \bar{\Psi}^a\sigma^\mu D_\mu\Psi_a - C^{ab}{}_m\Psi_a\left[\Phi^m,\Psi_b\right] - \bar{C}_{mab}\,\bar{\Psi}^a\left[\Phi^m,\bar{\Psi}^b\right]\right] \tag{2.15}$$

with $g_{\text{YM}}^2 = 2\pi g_s$. This is a conformal quantum field theory for any value of N and constant g_{YM}^2.

- **Low-energy limit from the supergravity picture**. In this case we also obtain two decoupled sectors, but the limit is slightly more subtle. Indeed, if we take the limit $\ell_s \to 0$ in Eq. (2.10) it would seem, considering Eq. (2.13), that we are left just with flat space. This is true everywhere except in a small region close to the origin at $r=0$, where the metric becomes

$$\text{d}s_5^2 = \frac{r^2}{L^2}\,\eta_{\alpha\beta}\text{d}x^\alpha\text{d}x^\beta + \frac{L^2}{r^2}\text{d}r^2 + L^2\text{d}\Omega^5\,. \tag{2.16}$$

[1]More precisely, the gauge group is $U(N)$; the trace $U(1)$ is the free center-of-mass multiplet of the stack and decouples, so we drop it.

This is $AdS_5 \times S^5$. Therefore, in the limit we are considering, the geometry can be thought of as ten-dimensional flat Minkowski space "pasted" to a long AdS throat at $r \simeq 0$.

What is then the low-energy description of this system? On the one hand, we obtain again a sector of (low-energy) closed strings propagating in ten-dimensional flat space. In addition, we have the *full* Type IIB closed string theory propagating in $AdS_5 \times S^5$: in this sector we get the full string theory as modes propagating and interacting in the throat, which are still low-energy from the perspective of an asymptotic observer, since these modes have to escape a gravitational potential to escape the throat and thus will be red-shifted.

Note that in the low-energy limit both descriptions lead to a decoupling into two different sectors. Since one of these sectors coincides on both sides, we are invited to conjecture that the other two sectors should be equivalent. Hence, we claim that

$\underline{\mathcal{N}=4\ \mathrm{SU}(N)\ \text{SYM in } d=4 \text{ dimensions}}$ ***is dual to*** $\underline{\text{Type IIB string theory in } \mathrm{AdS}_5\times \mathrm{S}^5}$.

We shall now match the parameters on both sides of the duality. We already stated that the YM coupling appearing in the Lagrangian Eq. (2.15) is related to the string coupling as $g^2_{\rm YM} = 4\pi g_s$. This description will be useful when we can perform perturbative expansions in terms of the 't Hooft coupling $\lambda = g^2_{\rm YM}N$, as we discussed in the previous section. Remarkably, on the gravity side we discover from Eq. (2.13) that

$$\frac{L^4}{\ell_s^4} = 4\pi g_s N = 2 g^2_{\rm YM} N = 2\lambda\,. \tag{2.17}$$

Note that gravity can be treated classically when $L \gg \ell_s$ to suppress stringy effects and $g_s \ll 1$ to suppress quantum gravity effects. This forces us to consider $\lambda \to \infty$ and also $N \to \infty$, faster than $g_s \to 0$. But note that this corresponds precisely to the strongly-coupled regime of the CFT. This is one of the major virtues of AdS/CFT: at large N it provides access to the strongly coupled physics of a gauge theory, relating its observables to a classical theory of gravity; conversely, it relates stringy effects in a theory of gravity to perturbative physics. Finite N effects incorporate quantum gravity effects in the bulk and non-planar diagrams on the gauge theory side.

Let us now observe some facts that support the conjecture. First, we can observe that the symmetries on both sides of the duality match. The gauge-theory side is invariant under $\mathrm{Conf}(3,1)\times \mathrm{SU}(4)$: the first factor corresponds to the Poincaré group plus four special conformal transformations

$$x^\mu \quad\mapsto\quad x'^\mu \;=\; \frac{x^\mu - b^\mu x^2}{1 - 2\, b\cdot x + b^2 x^2}\,, \tag{2.18}$$

and dilatations

$$x^\mu \quad\mapsto\quad x'^\mu \;=\; \Lambda x^\mu\,, \tag{2.19}$$

The second factor corresponds to the R-symmetry of $\mathcal{N}=4$ SYM. Note that these groups share the same Lie algebra structure as $\mathrm{Conf}(3,1) \simeq \mathrm{SO}(4,2)$ and $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$, which means that the global symmetry group on the gauge-theory side matches the isometry group of $\mathrm{AdS}_5 \times \mathrm{S}^5$ on the gravity side, $\mathrm{SO}(4,2)\times \mathrm{SO}(6)$.

While special conformal transformations (2.18) are slightly messy, dilatations (2.19) have a nice geometrical implementation in the bulk. Note that in AdS the corresponding isometry also contains

$$r \quad\mapsto\quad r' \;=\; \Lambda^{-1} r\,, \tag{2.20}$$

This means that the r direction in the bulk transforms as an energy scale, rather than a length. This means that we can interpret the holographic radial direction as a geometrization of the RG flow: high-energy phenomena and short-distance physics are mapped to the region close to the boundary (large values of r), while regions deeper into the bulk correspond to lower-energy physics on the gauge-theory side.