

3 Concrete computations: finite temperature and correlators

We have just learned that there seems to be a connection between gravity in $\text{AdS}_5 \times \text{S}^5$ and $\mathcal{N} = 4$ SYM in four dimensions. In this section we will start exploring how observables on both sides of the duality are connected. We begin by looking at solutions with horizons. In many cases, the five-sphere plays a spectator role, and we can actually work directly with the five-dimensional *truncated* action that we obtain after integrating over the sphere.¹

$$\begin{aligned} S_{\text{IIB}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}X \sqrt{-G_{[10]}} \left[R_{[10]} - \frac{1}{4 \cdot 5!} F_5^2 + \dots \right], \\ \mapsto S_{5\text{D}} &= \frac{1}{2\kappa_5^2} \int d^5X \sqrt{-g} (R - 2\Lambda), \end{aligned} \quad (3.1)$$

with R the Ricci scalar of the five-dimensional metric $g_{\mu\nu}$, and g its determinant. Notice, for instance, how the (negative) cosmological constant in the 5D action arises after integrating the flux F_5 over the five-sphere. Recall that the ten-dimensional Newton’s constant is $2\kappa_{10}^2 = (2\pi\ell_s)^8 g_s^2 / (2\pi)$, from which we obtain

$$\frac{1}{2\kappa_5^2} = \frac{\text{vol}(\text{S}^5)}{2\kappa_{10}^2} = \frac{L^5}{2^7 \pi^4 \ell_s^8 g_s^2}. \quad (3.2)$$

3.1 Finite temperature

In the previous section we saw how a stack of N coincident D3-branes can be described, from the gravity perspective, as a solitonic object with a certain mass. Now, if we had “too many” of such branes, we expect that they collapse into a black object. Indeed, the corresponding solution is

$$\begin{aligned} ds_{10}^2 &= H(r)^{-\frac{1}{2}} \left[-f(r) dt^2 + d\vec{x}^2 \right] + H(r)^{\frac{1}{2}} \left[f(r)^{-1} dr^2 + r^2 d\Omega_{(5)}^2 \right], \\ H(r) &= 1 + \frac{L^4}{r^4}, \quad f(r) = 1 - \frac{r_{\text{H}}^4}{r^4}. \end{aligned} \quad (3.3)$$

Actually, this is a black *brane*, rather than a black hole: the horizon is non-compact. In the decoupling limit, this metric becomes

$$ds_5^2 = \frac{r^2}{L^2} \left(-f(r) dt^2 + d\vec{x}^2 \right) + \frac{L^2}{r^2} \frac{dr^2}{f(r)}. \quad (3.4)$$

These objects have thermodynamic properties that we will examine next. For this reason it is natural to identify them as thermal states of the gauge theory—they describe strongly coupled, homogeneous $\mathcal{N} = 4$ plasmas. Let us see how to obtain their properties from standard black hole thermodynamics.

How do we study gravity at finite temperature? Recall first how it works in QFT. Given a theory whose Lagrangian density is \mathcal{L} , its path integral reads

$$Z = \int \mathcal{D}\phi \exp \left[i \int d^4x \mathcal{L} \right]. \quad (3.5)$$

¹In general, when going from the ten-dimensional action to a five-dimensional one we get (many) additional fields that come from the fluxes and from the part of the geometry we are integrating out. So this step should be interpreted as constructing a 5D action whose solutions are also solutions of the 10D theory, even though the 10D theory may have more solutions that we do not see in the reduction because we are setting some fields either to zero or to constants.

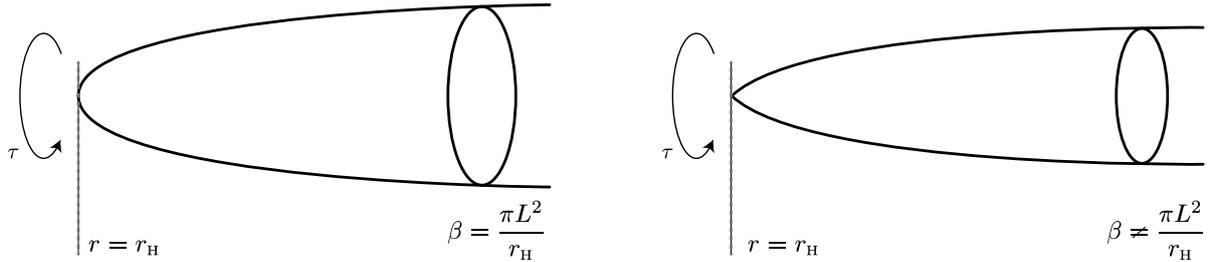


Figure 1: Behaviour of the Euclidean time circle near the horizon of the black brane. Regularity fixes the value of β , since a conical singularity would be present otherwise.

Then, the finite-temperature partition function can be obtained by Wick rotating the time direction $t \rightarrow -i\tau$,

$$\mathcal{Z} = \int \mathcal{D}\phi \exp \left[- \int_0^\beta d\tau \int d^3\vec{x} \mathcal{L}_E \right], \quad (3.6)$$

with $\beta = 1/T$.

In quantum field theory, the path integral is integrated over the fields, $\mathcal{D}\phi$. If we want to take a similar approach to quantum gravity, we would also need to allow for different configurations of spacetime, and integrate over all possible metrics, $\mathcal{D}g$. At finite temperature, we would then consider

$$\mathcal{Z}_{\text{QG}} = \int \mathcal{D}g_{\mu\nu} \exp[-S_E], \quad (3.7)$$

where, in our case, S_E should be the Euclidean version of Eq. (3.1) (including the necessary boundary terms), with compact Euclidean time.

Consider then the Wick-rotated version of Eq. (3.4), namely

$$ds_{\text{E}}^2 = \frac{r^2}{L^2} \left(f(r) d\tau^2 + d\vec{x}^2 \right) + \frac{L^2}{r^2} \frac{dr^2}{f(r)}, \quad (3.8)$$

with $\tau \in (0, \beta)$. Importantly, β cannot be arbitrary: it is fixed by regularity at the horizon. Indeed, if we perform the change of radial coordinate $r = r_{\text{H}}(1 + \rho^2/L^2)$, the metric close to the horizon becomes

$$ds_{\text{E}}^2 \simeq d\rho^2 + \frac{4r_{\text{H}}^2}{L^4} \rho^2 d\tau^2 + \frac{r_{\text{H}}^2}{L^2} d\vec{x}^2. \quad (3.9)$$

The (τ, ρ) part of the metric looks like flat space written in polar coordinates close to $r = r_{\text{H}}$. For that, we need that the angle about the origin $\rho = 0$ ranges from 0 to 2π , which fixes β ,

$$\frac{2r_{\text{H}}}{L^2} \tau \in (0, 2\pi) \quad \Rightarrow \quad \tau \in (0, \beta) = \left(0, \frac{\pi L^2}{r_{\text{H}}} \right). \quad (3.10)$$

In particular, the temperature is

$$T = \frac{r_{\text{H}}}{\pi L^2}. \quad (3.11)$$

From Eq. (3.9) it is also easy to compute the entropy, given in terms of the area of the horizon,

$$S = \frac{2\pi}{\kappa_5^2} \text{Area}(r = r_{\text{H}}) = \frac{2\pi}{\kappa_5^2} \frac{r_{\text{H}}^3}{L^3} \text{vol}(\mathbb{R}^3). \quad (3.12)$$

Writing now Newton's constant in terms of fundamental constants using Eq. (3.2) and recalling that $L^4/\ell_s^4 = 4\pi g_s N$, we obtain

$$S = \frac{1}{2} \pi^2 T^3 N^2 \text{vol}(\mathbb{R}^3) \quad (\text{at strong coupling}). \quad (3.13)$$

This is a remarkable result for many reasons. First, because we are computing the entropy of a black object, it scales with the area of the horizon (in five dimensions, the horizon area is proportional to $\text{vol}(\mathbb{R}^3)$). But now we understand why: the black brane is actually encoding the entropy of the dual plasma. Conversely, the black hole microstates are contained in the Hilbert space of strongly-coupled $\mathcal{N} = 4$ SYM.

Second, and related to the previous point, note that the entropy scales as $O(N^2)$. This reflects that the plasma is *deconfined* in the sense that it is constituted by colored gluons, whose degrees of freedom are represented by $N^2 - 1$ independent matrices. In a confined phase, in contrast, one expects the entropy to scale as N^0 (colored degrees of freedom are not excited).

Also, note that the weak-coupling result can be obtained by considering the black-body problem of a system of $6N^2$ scalars, $4N^2$ Weyl fermions, and the gauge field. The result is

$$S = \frac{2}{3}\pi^2 T^3 N^2 \text{vol}(\mathbb{R}^3) \quad (\text{at weak coupling}). \quad (3.14)$$

The discrepancy between Eqs. (3.13) and (3.14) is not problematic; rather it is telling us that there is a dependence on the coupling λ . In general, we expect the expression for an arbitrary value of λ to be

$$S = \frac{2\pi^2}{3} f(\lambda) T^3 N^2 \text{vol}(\mathbb{R}^3). \quad (3.15)$$

One can perform perturbative computations and find corrections at small coupling (see Refs. [1–4]),

$$f(\lambda) = 1 - \frac{3}{2\pi^2}(2\lambda) + \frac{3 + \sqrt{2}}{\pi^3}(2\lambda)^{3/2} + \dots \quad (3.16)$$

Similarly, as shown in Ref. [5], from the leading ℓ_s correction to the Type IIB action one can find

$$f(\lambda) = \frac{3}{4} + \frac{45}{32} \zeta(3) (2\lambda)^{-3/2} + \dots \quad (3.17)$$

It is conjectured that $f(\lambda)$ is a monotonically decreasing function of the 't Hooft coupling.

3.2 Absorption by branes

The computation we just made showed that the relation between observables on both sides of the duality can be made precise. But this goes beyond thermodynamics, as we shall argue next. More precisely, we would like to understand how the main characters in the CFT — n -point correlation functions— can be computed on the gravity side. Relatedly, we would like to understand how the path integral of $\mathcal{N} = 4$ SYM is related to the gravitational path integral in AdS_5 .

To understand these relations, let us go back for a moment to the two available descriptions of the stack of N branes (without a horizon, at zero temperature). Consider then the following process: a closed string carrying a dilaton state interacts with the set of branes and splits into open strings. We can explore this process at low energies from the two available descriptions of the stack of branes:

- If we think of the branes as boundary conditions for open strings, at low energies we obtain (as we know) the $\mathcal{N} = 4$ Lagrangian. This is not enough because it does not contain the coupling between bulk and brane fields (i.e. closed and open strings). We therefore need the interaction term, coming from the next order in ℓ_s ,

$$S_{\text{int}} = T_3 \int d^4x \left[\text{tr} \left(\frac{1}{4} \Phi F_{\alpha\beta}^2 - \frac{1}{4} C_4 F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right) + \frac{1}{2} h^{\alpha\beta} T_{\alpha\beta} \right], \quad (3.18)$$

where $T_{\alpha\beta}$ is the $\mathcal{N} = 4$ energy-momentum tensor. Note that, from the perspective of the $d = 4$ Lagrangian, the bulk fields Φ , C_4 and $h^{\alpha\beta} = G_{\alpha\beta} - \eta_{\alpha\beta}$ act as external sources.

From this interaction we learn, for instance, that a particle arriving at the brane may be converted into a pair of gluons moving in opposite directions. After canonically normalizing the fields (which we indicate with tilded quantities), the relevant coupling is

$$-\sqrt{\frac{\pi}{2}} T_3^{-1} \int d^4x \tilde{\phi} \text{Tr}(\partial_\alpha \tilde{A} \partial^\alpha \tilde{A}). \quad (3.19)$$

The amplitude for the process is then, for each species (i.e. each of the gluons that can be radiated),

$$\mathcal{A} = -\sqrt{\frac{\pi}{2}} T_3^{-1} 2 \frac{p_1 \cdot p_2}{\sqrt{2\omega^{\frac{3}{2}}}} = -\frac{\sqrt{\omega\pi}}{2T_3}. \quad (3.20)$$

Each species hence contributes to the absorption cross-section with

$$\frac{1}{2} \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} (2\pi)^4 \delta(E_1 + E_2 - \omega) \delta^3(\vec{p}_1 + \vec{p}_2) \mathcal{A}^2. \quad (3.21)$$

With $E_1 = E_2 = \omega/2$, $\vec{p}_1 = -\vec{p}_2$ and $p_1 \cdot p_2 = \omega^2/2$, performing the integral and multiplying by $2N^2$ (the number of species) we get

$$\sigma = \frac{N^2 \omega^3}{32T_3} = \frac{1}{8} L^8 \pi^2 \omega^3, \quad (3.22)$$

where in the last equality we used the expression for the tension of the D3-branes and the flux quantization condition for F_5 .

- On the other hand, let us consider the same process from the supergravity perspective. In this case we ask for the absorption cross-section of the geometry. That is, we consider a dilaton wave incident from far away (i.e. the $r \gg L$ region), and we compare the incoming flux with the flux that is reflected. The equation we need to solve is just the Klein–Gordon equation

$$\nabla_\mu \nabla^\mu \Phi = 0, \quad (3.23)$$

with μ running over the ten coordinates. For a general field, Eq. (3.23) can be decomposed in spherical harmonics and plane waves along the gauge theory directions,

$$\Phi = e^{-i\omega t + i\vec{k} \cdot \vec{x}} Y_{l,n_1,n_2,n_3,m}(\theta_i) \phi(r), \quad (3.24)$$

with $Y_{l,n_1,n_2,n_3,m}(\Omega)$ the spherical harmonics on the S^5 . For the dilaton, the relevant equation is that of the s -wave channel without momentum dependence, $\Phi = e^{-i\omega t} \phi(r)$,

$$\left(\frac{d^2}{dr^2} + \frac{5}{r} \frac{d}{dr} + \omega^2 + \frac{L^4 \omega^2}{r^4} \right) \phi(r) = 0. \quad (3.25)$$

Let us analyse this equation both in the throat and in the asymptotic region. For the asymptotic region it is convenient to introduce the coordinate $\rho = \omega r$ and the redefinition $\phi(\rho) = \rho^{-5/2} \psi(\rho)$, in terms of which Eq. (3.25) reads

$$\left(\frac{d^2}{d\rho^2} - \frac{15}{4\rho^2} + 1 + \frac{(\omega L)^4}{\rho^4} \right) \psi(\rho) = 0. \quad (3.26)$$

For $\rho \gg (\omega L)^2$ we can ignore the last term, in which case the general solution is

$$\psi(\rho) = \sqrt{\rho}(a_1 J_2(\rho) + a_2 Y_2(\rho)) , \quad \Rightarrow \quad \phi(\rho) = \frac{1}{\rho^2}(a_1 J_2(\rho) + a_2 Y_2(\rho)) , \quad (3.27)$$

with J_n and Y_n the Bessel functions of the first and second kind respectively. Because we are interested in a solution that extends regularly to the overlap region, we set $a_2 = 0$, since $Y_2(\rho)$ diverges at $\rho = 0$.

To study the opposite regime, in contrast, we introduce the coordinate

$$z = \frac{(\omega L)^2}{\rho} \quad (3.28)$$

and the variable $f(z) = z^{-3/2}\phi(\rho(z))$. Then Eq. (3.25) becomes

$$\left(\frac{d^2}{dz^2} - \frac{15}{4z^2} + 1 + \frac{(\omega L)^4}{z^4} \right) f(z) = 0 . \quad (3.29)$$

In this case, we can again ignore the last term when $z \gg (\omega L)^2$. Note that this corresponds to $\rho \ll 1$. The general solution is then

$$f(z) = \sqrt{z}(b_1 J_2(z) + ib_2 Y_2(z)) , \quad (3.30)$$

and therefore

$$\phi(\rho) = i \frac{(\omega L)^4}{\rho^2} \left[b_1 J_2 \left(\frac{(\omega L)^2}{\rho} \right) + ib_2 Y_2 \left(\frac{(\omega L)^2}{\rho} \right) \right] . \quad (3.31)$$

Close to $z = 0$,

$$f(z) \propto \frac{1}{z^2} \left((b_1 - b_2)e^{-iz} (1 + O(z)) - (b_1 + b_2)e^{iz} (1 + O(z)) \right) . \quad (3.32)$$

Since we want an ingoing wave at the throat, we set $b_1 = b_2$ (the wave moves towards increasing values of z).

We can now compute the absorption ratio. The crucial observation is that the two solutions we have found, Eqs. (3.27) and (3.30),

$$\begin{aligned} \phi(\rho) &= \frac{a_1}{\rho^2} J_2(\rho) & \rho \gg (\omega L)^2 \\ \phi(\rho) &= ib_1 \frac{(\omega L)^4}{\rho^2} \left[J_2 \left(\frac{(\omega L)^2}{\rho} \right) + iY_2 \left(\frac{(\omega L)^2}{\rho} \right) \right] & \rho \ll 1, \end{aligned} \quad (3.33)$$

overlap in the region $(\omega L)^2 \ll \rho \ll 1$, which is sensible when ωL is small. We are therefore allowed to expand the solution in the second line for $\rho \gg (\omega L)^2$ and impose that it equals the first solution, which leads to

$$a_1 = \frac{32}{\pi} b_1 . \quad (3.34)$$

Now we compute the flux. For our KG field the conserved current is

$$j^M = \frac{1}{2i} \left(\Phi^* \nabla^M \Phi - \Phi \nabla^M \Phi^* \right) , \quad (3.35)$$

and the radial flux through an S^5 at fixed r is (up to the sphere volume)

$$\mathcal{F} = \int d\Omega_5 \sqrt{-G} j^r = i \frac{\pi^3}{2} r^5 (\phi(r) \phi'(r)^* - \phi(r)^* \phi'(r)). \quad (3.36)$$

In the asymptotic region, the wave has both incoming and outgoing parts,

$$\phi(\rho) = \frac{a_1}{\rho^2} J_2(\rho) \simeq \phi_{\text{in}}(\rho) + \phi_{\text{out}}(\rho) = -\frac{(16 + 16i)e^{-i\rho}}{\pi^{3/2}\rho^{5/2}} - \frac{(16 - 16i)e^{i\rho}}{\pi^{3/2}\rho^{5/2}} + \dots \quad (3.37)$$

Using $\phi_{\text{in}}(\rho)$ in Eq. (3.36) we obtain

$$\mathcal{F}_{\text{in}} = -\frac{512}{\omega^5}, \quad (3.38)$$

while the flux outgoing into the throat is

$$\mathcal{F}_{\text{out}} = -2L^8 \omega^3 \pi^2. \quad (3.39)$$

The quotient between these two fluxes gives the absorption probability, $\mathcal{P} = \mathcal{F}_{\text{out}}/\mathcal{F}_{\text{in}}$. The absorption cross-section is given in terms of the s -wave absorption probability by [6]

$$\sigma = \frac{(2\pi)^5}{\omega^5 \text{vol}(S^5)} \mathcal{P} = \frac{1}{8} L^8 \pi^2 \omega^3. \quad (3.40)$$

Note this coincides precisely with Eq. (3.22). The reason why in this case there is no mismatch is that this quantity is protected by supersymmetry.

3.3 Correlation functions and bulk/boundary correspondence

We have just seen that from the field-theory perspective, a wave that perturbs the brane is experienced as turning on an external coupling to one of the operators. More generally, closed-string (bulk) fields couple to gauge-invariant operators of the D3-brane worldvolume theory. At low energies the coupling has the schematic form

$$S_{\text{int}} = \int d^4x \phi(x, 0) \mathcal{O}(x), \quad (3.41)$$

where $\phi(x, 0)$ is the value of the bulk field evaluated at the brane (and, equivalently, the field-theory source), while \mathcal{O} is the operator it couples to.

In the near-horizon description this same statement is rephrased geometrically: the quantity $\phi(x, 0)$ is identified with the boundary data of the corresponding AdS field. Specifying this boundary value fixes the bulk solution, i.e. it determines how the field propagates into the throat. We conclude that deforming the CFT by a source is dual to solving the bulk equations with prescribed boundary conditions at the AdS boundary.

Let us now make this identification precise. In QFT, adding a source $J(x)$ for an operator $\mathcal{O}(x)$ defines the generating functional

$$Z_{\text{CFT}}[J] \equiv \left\langle \exp \left[\int d^4x J(x) \mathcal{O}(x) \right] \right\rangle \quad (3.42)$$

and the connected correlators follow from $W[J] = \log Z_{\text{CFT}}[J]$,

$$\langle \mathcal{O}(x) \rangle_J = \frac{\delta W[J]}{\delta J(x)}, \quad \langle \mathcal{O}(x) \mathcal{O}(y) \rangle_c = \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} \Big|_{J=0}, \quad (3.43)$$

and similarly for higher-point functions.

On the gravity side, the claim is that specifying boundary data for a bulk field is exactly the same operation: the boundary value plays the role of the source. More explicitly,

$$Z_{\text{CFT}}[J] = Z_{\text{SUGRA}}[\Phi|_{\partial} = J], \quad (3.44)$$

or more generally

$$Z_{\text{CFT}}[g^{(0)}, J] = Z_{\text{SUGRA}}[g \rightarrow g^{(0)}, \Phi \rightarrow J]. \quad (3.45)$$

In the classical regime (large N and large 't Hooft coupling) the gravitational path integral is dominated by a saddle point, so that

$$Z_{\text{SUGRA}}[\Phi|_{\partial} = J] \simeq \exp[-S_{\text{E,ren}}[\Phi_{\text{cl}}; \Phi|_{\partial} = J]], \quad (3.46)$$

where Φ_{cl} is the classical solution with the prescribed boundary condition and $S_{\text{E,ren}}$ is the Euclidean on-shell action including boundary terms and counterterms. Combining this with the QFT definition gives the practical prescription

$$W[J] \equiv \log Z_{\text{CFT}}[J] = -S_{\text{E,ren}}[\Phi_{\text{cl}}; \Phi|_{\partial} = J], \quad (3.47)$$

and therefore

$$\langle \mathcal{O}(x) \rangle_J = -\frac{\delta S_{\text{E,ren}}}{\delta J(x)}, \quad \langle \mathcal{O}(x)\mathcal{O}(y) \rangle_c = \frac{\delta^2 S_{\text{E,ren}}}{\delta J(x)\delta J(y)} \Big|_{J=0}. \quad (3.48)$$

and similarly for higher n -point functions.

The same logic applies to other bulk fields. For instance, the boundary metric $g_{\mu\nu}^{(0)}$ sources the stress tensor,

$$\delta W[g^{(0)}] = \frac{1}{2} \int d^4x \sqrt{g^{(0)}} \langle T^{\mu\nu} \rangle \delta g_{\mu\nu}^{(0)}, \quad (3.49)$$

and a bulk gauge field A_{μ} sources a conserved current J^{μ} . In all cases, the statement is that turning on boundary data for bulk fields corresponds to turning on sources for the dual operators in the CFT, and correlators follow by differentiating the renormalized on-shell action.

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