4 Applications

Now that we have made AdS/CFT precise, let us inspect some observables and see how gravity provides a geometric way to compute them at strong coupling.

4.1 Wilson loops

In Yang–Mills theories a powerful (non-local) operator that provides information about different phases of the theory is the Wilson loop. Consider again the YM part of the QCD Lagrangian,

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\, F^a_{\mu\nu} F^{a\,\mu\nu}\,, \tag{4.1}$$

with SU(N) gauge group. For sufficiently large N, the theory experiences a phase transition at some critical temperature T_c (for pure SU(3) YM, $T_c \simeq 270\mathrm{MeV}$). We shall discuss now how Wilson loops can distinguish between the two phases.

Consider first the Wilson line operator, defined along a path γ between x_0 and x_1,

$$U_{ij}[\gamma] \equiv \mathcal{P}\exp\Big(ig_{\mathrm{YM}}\int_\gamma \mathrm{d}x^\mu\, A^a_\mu(x)\, T^a\Big)_{ij}\,, \tag{4.2}$$

where $\mathcal{P}$ stands for *path ordering*. Note that U_{ij} is not gauge invariant by itself, since it carries fundamental color indices. Gauge invariance is restored once the endpoints are contracted with fields transforming in the fundamental/anti-fundamental representation. In particular, in a theory containing a very heavy quark Q in the fundamental representation, U_{ij} is naturally related to the heavy-quark two-point function $\langle Q_i(x_1)\,\bar{Q}_j(x_0)\rangle$.

To see this explicitly, add such a quark to the YM Lagrangian,

$$\mathcal{L}_{\mathrm{YM}\,+\,Q} = -\frac{1}{4}\, F^a_{\mu\nu} F^{a\,\mu\nu} + \bar{Q}\,(i\gamma^\mu D_\mu - M)\,Q, \qquad D_\mu = \partial_\mu - ig_{\mathrm{YM}} A^a_\mu T^a\,. \tag{4.3}$$

In the limit of infinitely massive quark, $M \to \infty$, the quark becomes a classical color source whose trajectory can be taken as prescribed. Let us parameterize this worldline by τ,

$$\gamma:\ \tau \mapsto x^\mu(\tau)\,, \qquad \dot{x}^\mu(\tau) \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,. \tag{4.4}$$

In this limit, the propagation of the color wavefunction along γ is governed by a first-order equation,

$$\begin{gathered}\Big(\frac{\mathrm{d}}{\mathrm{d}\tau} + M - ig_{\mathrm{YM}}\,\dot{x}^\mu(\tau)\, A^a_\mu(x(\tau))\, T^a\Big)\, Q(\tau) = \\ \Big(\dot{x}^\mu(\tau)\partial_\mu + M - ig_{\mathrm{YM}}\,\dot{x}^\mu(\tau)\, A^a_\mu(x(\tau))\, T^a\Big)\, Q(\tau) = 0\,,\end{gathered} \tag{4.5}$$

where $Q(\tau)$ is a fundamental color vector and we used $\mathrm{d}/\mathrm{d}\tau = \dot{x}^\mu\partial_\mu$. This is the heavy-quark limit of the Dirac equation projected along the worldline of the quark.

The solution of Eq. (4.5) is

$$\begin{aligned} Q_i(\tau) &= e^{-M\tau}\,\Big[\mathcal{P}\exp\Big(ig_{\mathrm{YM}}\int_0^\tau \mathrm{d}\tau'\,\dot{x}^\mu(\tau')\, A^a_\mu(x(\tau'))\, T^a\Big)\Big]_{ij}\, Q_j(0) \\ &= e^{-M\tau}\,\Big[\mathcal{P}\exp\Big(ig_{\mathrm{YM}}\int_\gamma \mathrm{d}x^\mu\, A^a_\mu(x)\, T^a\Big)\Big]_{ij}\, Q_j(0)\,, \end{aligned} \tag{4.6}$$

which is precisely the Wilson line factor multiplying the trivial mass dependence $e^{-M\tau}$. In particular, in Euclidean time and for a prescribed worldline γ from x_0 to x_1, we can write schematically

$$\langle Q_i(x_1)\,\bar{Q}_j(x_0)\rangle \;=\; e^{-MT}\,\Big\langle U_{ij}[\gamma]\Big\rangle \times Z_R\,, \tag{4.7}$$

where Z_R denotes a (scheme-dependent) normalization/renormalization factor.

From what we just observed, an open Wilson line is not gauge invariant: under a gauge transformation it transforms as

$$U(x_1, x_0) \;\to\; g(x_1)\, U(x_1, x_0)\, g^{-1}(x_0)\,, \tag{4.8}$$

so it carries fundamental indices (or, equivalently, depends on the gauge rotation at its endpoints). A gauge-invariant object is obtained if the curve is closed, since in that case the endpoint transformations cancel inside the trace. This defines the *Wilson loop*,

$$W[C] \equiv \frac{1}{N}\, \mathrm{Tr}\, \mathcal{P} \exp\Big(i g_{\mathrm{YM}} \oint_C \mathrm{d}x^\mu\, A^a_\mu(x)\, T^a \Big)\,, \tag{4.9}$$

where $\mathcal{P}$ stands for *path ordering.*

The Wilson loop provides, for instance, a way to extract the potential between two (infinitely heavy) color sources. Recall that the Euclidean two-point function of an operator $\mathcal{O}$ admits the spectral decomposition

$$\langle \mathcal{O}(T)\, \mathcal{O}^\dagger(0) \rangle = \sum_n \left| \langle n | \mathcal{O}^\dagger(0) | 0 \rangle \right|^2 e^{-E_n T}\,, \tag{4.10}$$

so that in the $T \to \infty$ limit the ground state dominates,

$$\langle \mathcal{O}(T)\, \mathcal{O}^\dagger(0) \rangle \simeq \left| \langle 0 | \mathcal{O}(0) | n{=}0 \rangle \right|^2 e^{-E_0 T}\,, \qquad (T \to \infty). \tag{4.11}$$

Now consider a gauge-invariant operator that creates a static $Q\bar{Q}$ pair separated by a distance r,

$$\begin{aligned} \mathcal{O}_r(t) &= \bar{Q}(t, \vec{r})\, U[l(\vec{r}, t)]\, Q(t, \vec{0}) \\ &= \bar{Q}_i(t, \vec{r})\, U_{ij}\, [l(\vec{r}, t)]\;\, Q_j(t, \vec{0})\,, \end{aligned} \tag{4.12}$$

where $l(\vec{r}, t)$ is a *spatial* path at fixed Euclidean time t from $(t, \vec{0})$ to $(t, \vec{r})$, included to make the operator gauge invariant. Note that in Euclidean Heisenberg picture $\mathcal{O}_r(T) = e^{HT} \mathcal{O}_r(0) e^{-HT}$, so in particular

$$\langle \mathcal{O}_r(T)\, \mathcal{O}_r^\dagger(0) \rangle = \Big\langle \bar{Q}(T, \vec{r}) U[l(\vec{r}, T)] Q(T, \vec{0})\; \bar{Q}(0, \vec{0})\, (U[l(\vec{r}, 0)])^\dagger\, Q(0, \vec{r}) \Big\rangle, \tag{4.13}$$

Now, in the heavy-quark limit, Eq. (4.13) becomes a contraction of four Wilson lines, giving a Wilson (rectangular) loop. Indeed, together with the spatial connectors at $t = 0$ and $t = T$, the contractions form a closed rectangular contour $C_\Box$ of size $r \times T$, and we have

$$\langle \mathcal{O}_r(T)\, \mathcal{O}_r^\dagger(0) \rangle \;=\; e^{-2MT} \Big\langle W[C_\Box] \Big\rangle \times Z_R\,, \tag{4.14}$$

where Z_R denotes a (scheme-dependent) renormalization factor, see Eq. (4.7). Comparing (4.14) with (4.11) we conclude that, for large T,

$$\langle W[C_\Box] \rangle \sim e^{-E_0(r)\, T}\,, \tag{4.15}$$

where $E_0(r)$ is the lowest energy of the system in the presence of two static sources separated by r. It is therefore natural to identify this quantity with the potential energy between the two quarks. Hence, we define the static potential by this ground-state energy (up to an additive constant), yielding

$$V_{qq}(r) = -\lim_{T \to \infty} \frac{1}{T} \ln \langle W[C_\Box] \rangle\,. \tag{4.16}$$

Computing the Wilson loop is complicated in some situations, such as at strong coupling. For instance, if one is interested in computing the potential between quarks in pure glue SU(3) YM one needs to rely on lattice computations; see Fig. 1, adapted from Ref. [1]. From the figure, we see that the potential between quarks captures very nicely the difference between a confined[1] ($T = 0$) and a deconfined phase ($T > T_c$). On the one hand, in the confined phase the potential grows linearly for large separations and, in fact, manifests the Regge behaviour we discussed in the first lecture —the slope of the potential is nothing but the string tension. On the other hand, at finite temperature the potential flattens at a certain distance set by the temperature. This is a manifestation that the quarks are screened by the surrounding gluon plasma: intuitively, quarks "do not see each other" because there are too many gluons around scattering off them, if they are sufficiently separated.

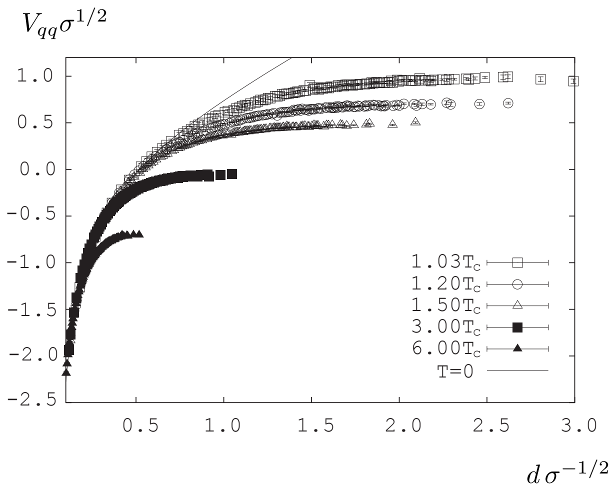

Figure 1: Quark-antiquark potential as a function of the separation d, at different temperatures in pure SU(3) Yang–Mills theory. The units are given in terms of the string tension at zero temperature, σ (i.e. the Regge slope). Adapted from Ref. [1].

4.2 Wilson loops in holography

Can we compute Wilson loops holographically, and in particular calculate something like the quark-antiquark potential of strongly coupled theories? It is actually possible. Indeed, we will argue that $\langle W[C_\square]\rangle$ can also be computed from some object in the gravity side, with approppriate boundary conditions.

To see how this goes, we need first to understand what is the dual in the gravity side to an infinitely massive quark. If we think again of the stack of branes to which open strings can end, we remember that we obtained a low energy limit in which all the states were massless, which can be thought of intuitively as coming from the fact that the strings want to minimize their length and, if the endpoints are siting on coincident branes, the least energetic state should have length zero (the string shrinks completly). This picture changes if one of the branes is separated from the rest. On the one hand, this gives a mass to the strings which have one endpoint on the separated brane and the other endpoint still ending on one of the branes in the stack. On the other hand, the gauge group is broken

$$\mathrm{U}(N) \longrightarrow \mathrm{U}(N-1) \times \mathrm{U}(1)\,. \tag{4.17}$$

Effectively, separating the brane has broken the gauge group and given a mass to the corresponding gauge fields —this is the reason why this procedure is known as *Higgsing*, in analogy with the electroweak Higgs mechanism. More precesily, we are giving a mass to the scalar field $|\vec{\phi}|$, as the set of fields ϕ^I described fluctuations of the branes in the transverse directions. Hence, we see that inserting a heavy particle (we can think of the endpoints ending on branes as quarks) corresponds to pushing the brane to infinity.

How does this mechanism look like in the gravity side (i.e. in $\mathrm{AdS}_5\times\mathrm{S}^5$)? Think of the stack of branes sitting at $r = 0$ and sourcing the AdS_5 geometry. If we separate one of them *a little*

[1] Watch out, here T is temperature while before it was time, sorry about that.

bit, so that it is still in the throat region, we are effectively including dynamical quarks in the game: now we can have *classical* open strings ending on the brane. With this, for instance, you could study mesons (see Ref. [2]). But here we are interested in Wilson loops, and we need to send the mass to infinity: the brane must scape the throat region. We conclude that a Wilson loop $\langle W[C_\square]\rangle$ is given by the length of the string whose endpoints are attached at the boundary of AdS to the loop $C_\square$ of the quarks.

Let us make it concrete, by computing the potential between probe charges in $\mathcal{N}=4$ SYM at strong coupling, as it was originally done in Ref. [3]. In $\mathcal{N}=4$ SYM, matter transforms in the adjoint representation, but after Higgsing the corresponding massive gauge boson W transforms in the fundamental representation, and in the $|\vec{\phi}|\to\infty$ limit they satisfy the equation of motion

$$\left(\partial_0 - iA_0 - i\,\theta^I X^I\right)\tilde{W} \;=\; 0\,, \tag{4.18}$$

where the time dependence $W = e^{-i|\vec{\phi}|t}\tilde{W}$ has been extracted and $\theta^I \equiv \phi^I/|\vec{\phi}|$. Hence, they provide the "infinitely" massive quarks necessary for the computation of the Wilson loop,

$$W[C] = \frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\exp\!\left(i\oint_C \mathrm{d}s\,\Big[A_\mu(x(s))\,\dot{x}^\mu(s) + \theta^I(s)\,X^I(x(s))\,|\dot{x}(s)|\Big]\right). \tag{4.19}$$

Note also the extra coupling to the fields X^I, which correspond to the transverse fluctuations of the brane. This coupling arises because the string endpoint is not only a source of gauge field A_μ on the brane, but also of "scalar" charge for the fields X^I. Now, each point in the loop is specified by the contour C, which we denote $x^\mu(s)$, and a path in S^5, namely $\vec{\theta}(\sigma)$. For simplicity we ser $\vec{\theta}(\sigma)$ to constant.

From everything we said so far, it is natural to propose that the expectation value of the Wilson loop is

$$\langle W[C]\rangle \propto \exp(-S) \tag{4.20}$$

and —in the large N and large $g_s N$ limit— S is the proper length of a string which is attached at C at the AdS boundary and lies along $\theta^I(s)$, given by the NG action

$$S_{\mathrm{NG}} = -\frac{1}{2\pi\ell_s^2}\int \mathrm{d}\tau\mathrm{d}\sigma\sqrt{-\det\left(G_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\right)}\,. \tag{4.21}$$

We will see that Eq. (4.25) is divergent and requires renormalization.

Let us the consider the rectangular Wilson loop. As we mentioned, we take the string to be sitting at a point in the S^5, so that $\theta^I(s) = \theta^I_0 =$ constant. For this reason the five-sphere does not play any role in what follows and we can forget about it and we can restrict our attention to the AdS factor of the geometry, which we write in the $z = 1/r$ coordinate,

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}z^2\right) \tag{4.22}$$

To find the potential between a quark and an antiquark that are separated a distance d, we just need to find the embedding of a string whose endpoints follow the worldlines

$$z = 0\,, \qquad t\in(0,T) \qquad x_1 = \pm\frac{d}{2}\,, \qquad x_2 = x_3 = 0\,. \tag{4.23}$$

For that, we can identify in the NG action (4.25)

$$t = \tau\,, \qquad x_1 = \sigma \equiv x\,, \qquad z = z(x)\,, \qquad x_2 = x_3 = 0\,. \tag{4.24}$$

and we obtain

$$
\begin{aligned}
S_{\cup} &= -\frac{1}{2\pi\ell_s^2}\int_0^T \mathrm{d}t \int_{-d/2}^{d/2} \mathrm{d}x\, \frac{L^2}{z(x)^2}\sqrt{1+z'(x)^2} \\
&= -\frac{2T}{2\pi\ell_s^2}\int_0^{d/2} \mathrm{d}x\, \frac{L^2}{z(x)^2}\sqrt{1+z'(x)^2} \\
&\equiv -\frac{2T}{2\pi\ell_s^2}\int_0^{d/2} \mathrm{d}x \mathcal{L}\left(z(x), z'(x)\right)\,.
\end{aligned}
\tag{4.25}
$$

Note that the integrand does not depend explicitely on x. This impliues that there is a conserved quantity Q along $z(x)$,

$$
\mathsf{Q} = z'(x)\frac{\partial\mathcal{L}}{z'(x)} - \mathcal{L} = -\frac{L^2}{z(x)^2\sqrt{z'(x)^2+1}} = -\frac{L^2}{z_*^2}\,, \tag{4.26}
$$

where in the last equality we used the boundary condition that at the turning point $z(0) = z_*$, $z'(0) = 0$. From this we can find a first integral of the equations of motion in terms of the position of the turning point z_* ,

$$
z'(x) = \pm\frac{\sqrt{z_*^4 - z(x)^4}}{z(x)^2}\,. \tag{4.27}
$$

We choose the "$-$" branch since we are in the $x \in (0, d/2)$ branch, where z decreases as x increases (since we are approaching the boundary of AdS). Using Eq. (4.27) we can write $\mathrm{d}x = z'(x)\mathrm{d}z$, therefore

$$
d = 2\int_0^{d/2}\mathrm{d}x = 2\int_0^{z_*}\mathrm{d}x\frac{z^2}{\sqrt{z_*^4 - z^4}} = \frac{2\sqrt{\pi}\,\Gamma\left(7/4\right)}{3\,\Gamma\left(5/4\right)}z_*\,. \tag{4.28}
$$

Similarly,

$$
S_{\cup} = -\frac{2T}{2\pi\ell_s^2}\int_0^{z_*}\mathrm{d}z\frac{L^2 z_*^2}{z^2\sqrt{z_*^4 - z^4}} \tag{4.29}
$$

This expression is nevertheless divergent. We regulate it by subtracting the length of two strings that follow the same worldlines but fall straight into the bulk. Then the integral

$$
\Delta S = S_{\cup} - 2S_{|} = -\frac{2T}{2\pi\ell_s^2}\left[\int_0^{z_*}\mathrm{d}z\frac{L^2}{z^2}\left(\frac{z_*^2}{\sqrt{z_*^4 - z^4}} - 1\right) - \int_{z_*}^{\infty}\mathrm{d}z\frac{L^2}{z^2}\right] = \frac{TL^2}{\pi\ell_s^2}\times\frac{\sqrt{\pi}\Gamma\left(3/4\right)}{z_*\Gamma\left(1/4\right)}\,. \tag{4.30}
$$

Going back to the relation between the Wilson loop and the potential between charges, we identify $\Delta S = TV_{qq}$. We conclude that the potential between two charges in strongly coupled $\mathcal{N} = 4$ SYM is

$$
V_{qq}(d) = -\frac{4\sqrt{2}\pi^2}{\Gamma\left(1/4\right)^4}\times\frac{\lambda^{\frac{1}{2}}}{d}\,. \tag{4.31}
$$

Note that at weak coupling we have $V_{qq} \propto \lambda/d$.

We can also ask what is the potential between quarks in the presence of a plasma, by repeating the computation in the black brane geometry

$$
\mathrm{d}s^2 = \frac{L^2}{z^2}\left(-f(z)\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \frac{\mathrm{d}z^2}{f(z)}\right)\,, \qquad f(z) = 1 - \frac{z^4}{z_{\mathrm{H}}^4}. \tag{4.32}
$$

Repeating the previous steps we arrive at

$$V_{qq} = \frac{1}{\pi \ell_s^2}\left[\int_0^{z_*} \mathrm{d}z \frac{L^2}{z^2}\left(\frac{z_*^2}{z_{\mathrm{H}}^2}\sqrt{\frac{z^4 - z_{\mathrm{H}}^4}{z^4 - z_*^4}} - 1\right) - \int_{z_*}^{z_{\mathrm{H}}} \mathrm{d}z \frac{L^2}{z^2}\right] = \\ = (2\lambda)^{\frac{1}{2}} T\left(1 - \frac{25\pi^{3/2}}{128\sqrt{2}\,\Gamma(9/4)^2} \times \frac{(1-\zeta_*^4)}{\zeta_*}\, {}_2F_1\left(\frac{1}{2}, \frac{3}{4}; \frac{1}{4}; \zeta_*^4\right)\right), \tag{4.33}$$

where we have used $z_{\mathrm{H}} = L^2 r_{\mathrm{H}}^{-1} = (\pi T)^{-1}$ and defined $\zeta_* = Z_*/z_{\mathrm{H}}$. Similarly

$$d = \frac{1}{T} \times \frac{2\sqrt{\pi}\Gamma(7/4)}{3\Gamma(5/4)} \times \zeta_* \sqrt{1-\zeta_*^4}\, {}_2F_1\left(\frac{1}{2}, \frac{3}{4}; \frac{5}{4}; \zeta_*^4\right). \tag{4.34}$$

Unfortunatedly in this case we cannot isolate $V_{qq}(q)$ explicitedly. But we can still plot them together parametrically, the results is shown in Fig. 2. Importantly, we discover that the function $V_{qq}(d)$ is not single-valued. Instead, it revelops a cusp at $d \simeq 0.277T^{-1}$. The interpretation for this behavior is that the string wants to break. Indeed, remember that V_{qq} is actually the length of the string connecting the two timelike Wilson lines minus the contribution of two straight lines that fall straight from the boundary to the horizon. Even though we introduced this *disconnected* configuration to regularize the *connected* one, it is a valid configuration (as it also solves the equations of motion deduced from Eq. (4.25). In particular, for separations such that $V_{qq}(d) > 0$ they are the preferred configuration, and for $d \simeq 0.277T^{-1}$ they are the only available configuration. This means that there is a phase transition between connected and disconnected strings: it becomes favored for the string to fall into the horizon and break.

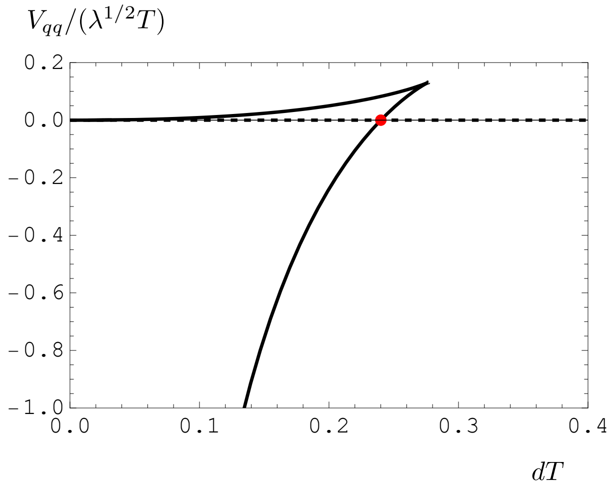

Figure 2: Quark-antiquark potential as a function of the separation d in the $\mathcal{N} = 4$ plasma, in units of the temperature. The solid curve corresponds to the "connected" configuration, while the dashed line stands for the "disconnected" ones. There is a phase transition between then when the curve cross each other.

4.3 Hydrodynamics

So far, we have seen how holography is capable of describing *thermodynamics* (i.e. equilibrium states) of a given quantum field theory by means of dual gravitational geometries with a horizon. But in many physical systems we can do more: we can near-equilibrium processes. This is the real of *hydrodynamics*, long-distance description for a given many-body system —see Ref. [4]. In hydrodynamics, the equations of motion express conservation laws of any quantity that is conserved. We focus in the energy momentum tensor,

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{4.35}$$

In general, $T^{\mu\nu}$ is a complicated object, which knows about the distribution of energy and stresses within a system. However, we experience quite often that the microscopic degrees of freedom are not relevant for may purposes. For example, to study the flow of water we need not track the trajectory of every single molecule as a function of time. Instead, the physics is well captured in terms of few variables suc has density, temperature, velocity of the fluid... The reason is that high energy (microscopic) processes withing the fluid *dissipate quickly*, leading to long-length propagating excitations. Whenever this is the case, we hope that conserved quantities can be expanded in gradients of slowly varying functions such as the fluid velocity $u^\mu(x)$, schematically,

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} \; + \; T^{\mu\nu}_{(1)} \; + \; T^{\mu\nu}_{(2)} + \cdots \,, \qquad T^{\mu\nu}_{(n)} \sim \mathcal{O}(\nabla^n)\,. \tag{4.36}$$

The different $T^{\mu\nu}_{(i)}$ are constructed from the built from all allowed tensors at each order. For instance, at leading order we have the ideal contribution (constructed from equilibrium configurations, no gradients),

$$T^{\mu\nu}_{(0)} = T^{\mu\nu}_{\text{ideal}} = (e+p)u^\mu u^\nu + p\, g^{\mu\nu}\,, \tag{4.37}$$

with e the enrgy density, and p the pressure, related by the equation of state $p(e)$ (note this can be constructed in general for a curved manifold with metric $g_{\mu\nu}$, as in Cosmology, though in many situations we restrict ourselves to the flat case $g^{\mu\nu} = \eta^{\mu\nu}$). At the next order, we encounter the first viscous hydrodynamic corrections,

$$T^{\mu\nu}_{(1)} = -\,\eta\,\sigma^{\mu\nu} - \zeta\,\Delta^{\mu\nu}\,(\nabla^\alpha u_\alpha)\,. \tag{4.38}$$

In this expression, η is the shear viscosity, which measures measures how strongly the fluid resists shape changes at (locally) fixed volume; and ζ is bulk viscosity, which measures how strongly the fluid resists uniform expansion or compression (i.e. a volume change). We also introduced the projector orthogonal to the fluid velocity,

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu\,, \tag{4.39}$$

and the shear tensor

$$\sigma^{\mu\nu} \equiv 2\,\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_{(\alpha}u_{\beta)} - \frac{1}{3}\,\Delta_{\alpha\beta}\,\nabla^\gamma u_\gamma\right)\,. \tag{4.40}$$

Thus, the terms in $T^{\mu\nu}_{(1)}$ contain only first derivatives of the velocity field u^μ.

In this section we examine how this hydrodynamic description is also captured holographically for an $\mathcal{N}=4$ plasma. Conformal invariance forces the stress energy momentum tensor to be traceless, which in particular implies

$$p = \frac{1}{3}e\,, \qquad \zeta = 0\,. \tag{4.41}$$

So everything that is left to determine at the linear order viscous correction is the shear viscosity η. This is given in terms of the two point function of the energy-momentum tensor throught the *Kubo formula*

$$\eta = \lim_{\omega\to 0}\frac{1}{2\omega}\int \mathrm{d}t\,\mathrm{d}^3\vec{x}\,e^{i\omega t}\,\langle[T_{xy}(t,\vec{x}),T_{xy}(0,0)]\rangle = -\lim_{\omega\to 0}\frac{1}{\omega}\,\mathrm{Im}\,G^R(\omega,0)\,, \tag{4.42}$$

which is obtained by introducing a tiny, time-dependent shear deformation of spacetime δh_{xy}: on the one hand, linear response says the induced shear stress is controlled by the stress–tensor correlator (with the retarded choice enforcing causality), while on the other hand hydrodynamics says the same deformation produces a viscous stress proportional to the shear rate, so matching the two descriptions in the low-frequency limit isolates η.

The nice thing of Eq. (4.42) is that it contains the two-point function of the energy-momentum tensor, which we know how to compute using holography. In the exercise today you will show that, for rather generic five-dimensional black brane solutions,

$$\frac{\eta}{s} = \frac{1}{4\pi}\,, \tag{4.43}$$

with s the entropy density.

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